Introducing risk into a Tobin asset-allocation model

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September 2017

Post Keynesian Economics Study Group

Working Paper 1713

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Abstract: The Tobin asset-allocation model has become a standard component in stock-flow consistent (SFC) models. It relates asset returns to wealth allocation, and thereby to the value of assets as reflected in Tobin's q. The model is flexible, parsimonious, and intuitively appealing, but it suffers from a large number of independent coefficients and depends only on returns for the allocation. A truism from financial theory and practice is that allocations depend on both risk and return. In this paper we introduce risk into a Tobin model. We propose that allocations are a compromise between competing goals of low turnover and high return, constrained by the degree of risk that investors are willing to tolerate. In our model, the Tobin coefficients depend on asset-specific risk and a small number of independent parameters. The model also yields an expression for the q values of different assets as a function of risk and parameters reflecting market sentiment.

Keywords: Tobin formula, asset allocation, risk

JEL classifications: G11, D8, E12

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1 Introduction

Tobin (1969) brought monetary theory closer to financial theory by introducing a model containing a variety of assets, not just the loans and bonds of the models of the time. From a Keynesian perspective, he broadened the concept of liquidity preference to the general one of asset allocation. In Tobin's framework, asset allocations across all investors vary in response to returns, with positive own-responses and negative cross-responses. Those allocations determine market capitalization, which, divided by the value of the physical capital stock, gives the well-known Tobin q ratio. Because an overvaluation reduces long-run expected returns, returns to investors are inversely related to q. Investor behavior thus sets up a dynamic that drives q toward an equilibrium value, presumably equal to one.

Tobin's model is elegant, and in a linearized form has become a standard feature in stock-flow consistent (SFC) models (Godley and Lavoie 2007). Yet, it suffers from some weaknesses. Aside from some summing-up conditions, the coefficients are unconstrained, and while they can in principle be estimated from data, the number of free parameters is large, growing with the square of the number of assets or asset classes in the linearized model. As noted by Tobin himself (1969, 29), the model ignores influences on investment allocation other than returns, particularly risk. In this paper we address these weaknesses by introducing risk into the Tobin model. We argue that the coefficients should depend on risk, and provide an explicit formula for practical applications. The number of free parameters in our model is small, and they have a clear economic interpretation.

Following Knight (1921) and Keynes, we distinguish risk from uncertainty. The model assumes that investors agree on how risky an asset is by some conventional measure, such as its historical idiosyncratic volatility. Uncertainty is captured in the attitudes of investors towards risk. In a recession, or after a bursting bubble, investors are assumed to lower their target risk, while they raise it when the economy is performing well. With the model developed in this paper we find that this behavior drives investors towards liquidity in a down market and away from liquidity when the market is up.

2 The linearized Tobin model

Tobin sought to explain why the distribution of market capitalization should differ from the distribution of the underlying capital stock, and gave an answer in terms of expected returns. His model applies to sectors or very broad asset classes, rather than specific assets. We refer to these clusters of assets as "assets" for simplicity.

In linearized form, and normalized by total wealth, the Tobin asset allocation model is

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \vdots \\ \lambda_{n0} \end{pmatrix} + \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}, \tag{1}$$

where s_i is the amount of wealth held in asset i as a share of total wealth; r_i is real expected return on investment; λ_{i0} is the share when all assets have the same return, which we call the "base share"; and the λ_{ij}, with $i,j = 1,\ldots, n$ are coefficients expressing marginal substitution between different assets when returns change. The base shares satisfy

$$\sum_{i=1}^{n} \lambda_{i0} = 1, \tag{2}$$

while the λ_{ij} satisfy

$$\sum_{i=1}^{n} \lambda_{ij} = 0,\ j > 0. \tag{3}$$

Together, these criteria ensure that both the observed shares s_i and the base shares do indeed act like shares, by adding up to one. They should also be nonnegative, which is not guaranteed in the linear model, and must be checked in practical applications. The condition that the observed shares equal the base shares when all assets have the same return means that

$$\sum_{j=1}^{n} \lambda_{ij} = 0. \tag{4}$$

To reduce the number of free parameters, the conditions in equations (3) and (4) are sometimes ensured by imposing one summation condition and requiring that the coefficients be symmetric,

$$\lambda_{ij} = \lambda_{ji}. \tag{5}$$

For this paper we adopt the convention that the final asset, asset n, is money. To better reflect Tobin's original paper and focus on q, we assume that all other assets are backed by physical capital stocks. Out of total wealth W, $s_i W$ is held in asset i, so for physically-backed assets we can calculate q_i as

$$q_i = \frac{s_i W}{K_i}, \tag{6}$$

where K_i is the monetary value of the capital stock associated with asset i. Tobin assumed that capital was valued at replacement cost, but we argue that the relevant accounting convention should apply. To the extent that firms use historical cost to value fixed capital investment, as Lee (1994) claims, q's estimated using replacement cost will be too low. If a fraction m of total wealth is held as money, then the average value of q that applies to the whole economy is given by

$$\bar{q} = (1-m)\frac{W}{K}, \quad K = \sum_{i=1}^{n-1} K_i. \tag{7}$$

We can use this expression to substitute for W in equation (6), and solve for the asset shares in terms of the q's, as

$$s_i = (1-m)\frac{K_i}{K}\frac{q_i}{\bar{q}}, \quad i = 1, 2, \ldots, n-1, \tag{8}$$

$$s_n = m. \tag{9}$$

In the subsequent discussion we represent the values of the s_i when each q_i is equal to one by the symbol w_i, so

$$w_i = (1-m^*)\frac{K_i}{K}, \quad i = 1, 2, \ldots, n-1, \tag{10}$$

$$w_n = m^*. \tag{11}$$

Here, m^* is the desired money holding when $q_i = 1$ for every asset.

3 Conceptual discussion

With Tobin's basic model and our notation in hand, we introduce some concepts to motivate the model before we present it. As noted in the introduction, we seek to distinguish assets in terms of risk. To clarify our arguments, we first present a model with indistinguishable assets, and then discuss risk.

3.1 A Tobin model with indistinguishable assets

We begin with an economy in which the only distinction between assets is the returns they offer. In that case, it makes sense for an individual investor to put all of her wealth into the asset that offers the highest return; but that strategy is self-defeating at the macroeconomic level. As funds flow into the most attractive asset, its market price rises and its expected return falls, until another asset becomes more attractive. Astute investors will realize that this process is unproductive; over the long run, a buy-and-hold strategy would have been simpler and would give returns that track the underlying market value over the long run. At equilibrium, we expect returns to be equalized and q's to equal one, so the equilibrium value for the shares is w_i. With all returns equal, the shares are equal to the base shares, so we have

$$\lambda_{i0} = w_i. \tag{12}$$

Next, we propose a hypothetical initial state in which the allocation is at its equilibrium and all assets yield the same return. Subsequently, the expected return to asset j rises slightly, by an amount Δr_j. If all investors have fully embraced a buy-and-hold strategy, then the coefficients λ_{ij} are zero, and the allocation will not change in response to the higher expected returns. However, such an extreme policy is unrealistic. We consider what a reasonable adjustment might look like.

Given the self-defeating consequences of shifting too much wealth into a single asset with above-average returns, the absorptive capacity of the asset should be considered. This is given by its equilibrium share, w_j. Because the other assets have the same return and are otherwise indistinguishable, the only way to prioritize them is by absorptive capacity. Accordingly, we assume that as investors shift funds toward asset j, they remove funds from other assets in proportion to their holdings. This leads us to propose

$$\lambda_{ij} = a\left(\delta_{ij} w_i - w_i w_j\right), \tag{13}$$

where δ_{ij} is the Dirac function, equal to one when i=j and zero otherwise. These coefficients are symmetric and they satisfy the summing-up conditions in equations (3) and (4). The factor a is a measure of how much investors weight high returns over low turnover when allocating their wealth.

3.2 Risk

In our model we distinguish assets by their risk, as perceived or measured by investors. A large number of risk measures is available to investors and portfolio managers (Schulmerich, Leporcher, and Eu 2015), each reflecting a different aspect of risk. Broadly, risk is seen as the tendency for returns to differ from an expectation, whether of a historical mean for a particular asset or the return for a benchmark asset. These may apply to an individual asset – an absolute measure – or to the performance of one asset compared to the benchmark – a relative measure. Absolute measures are typically statistical measures of dispersion, such as standard deviation, variance, coefficient of variation, or volatility. This is a venerable way to define risk in financial economics, as it was used by Markowitz (1952) in his paper introducing modern portfolio theory and by Sharpe (1964) when he introduced the Capital Asset Pricing Model (CAPM). Relative measures are more diverse, including differences between average returns, correlations in returns between two assets, and relative cumulative probabilities (stochastic dominance). When a common benchmark is adopted and applied to all assets, such as the rate on long-term government bonds, a relative measure effectively becomes an absolute measure, so in practice many of these metrics produce values that are specific to each asset independently of the others, aside from a common benchmark.

While the metrics introduced by academic financial economists are widely used in practice, even by large firms, they are not without problems. Broadly-held expectations drawn from theory, such as a positive correlation between return and volatility, or the capital market line as an upper bound on returns, fail basic empirical tests (Thompson et al. 2006), and neoclassical financial economics is rife with "anomalies" and "puzzles" in search of an explanation. To be fair, the models have been adjusted to account for some of the failures, and the validity of the tests is disputed. Also, portfolios that perform as well as the market in the long run can be constructed by intelligently combining standard metrics with analysis of fundamentals and a healthy grasp of the difference between uncertainty and risk (Weigand 2014).[1]

Nevertheless, even with the modifications, the standard risk measures can be questioned. Bunn and Campbell (2015), professional investment managers, strongly recommend against the standard academic metrics. They argue that the only meaningful definition of risk is the probability of falling below a minimum target return for an entire portfolio. That is, risk inheres in the portfolio, not in individual assets, and depends on the financing goal of the institution or individual. They further argue that volatility per se is uninteresting because most returns distributions are skewed, anything above the minimum is good, and everything below the minimum is bad. As they point out, putting all funds into a low-volatility asset with a highly-predictable return of 6% per annum is a risky strategy for an organization that needs its holdings to grow at 7.5% per annum (Bunn and Campbell 2015, 63). They also entirely reject correlation as a measure of risk, and low or negative correlation as a measure of diversification, arguing that true diversification should be carried out along three axes: 1) information stream, 2) execution and 3) unique risks (Bunn and Campbell 2015, 84). That is, the assets should be priced in markets that respond to different information, the response to that information should be specific to that market or asset, and the asset should be exposed to different risks from the other assets in the portfolio. However, Bunn and Campbell are arguing against dominant practice, as they make clear in their book. We therefore cannot take their recommended practice as typical.

Based on the foregoing, we follow conventional academic economics, and dominant current practice, by assuming that investors assign each asset i a risk value v_i independent of other assets (aside from a common benchmark, which we leave unspecified). Beyond that assumption we are agnostic – the measure can be any absolute measure or a relative measure against a common benchmark. We also assume no relationship between risk and return – the correlation between them can be positive, negative, or zero.

4 Presentation of the model

In the model we characterize the market by two parameters: an overall tolerance for risk, v^*, and a preference for keeping turnover low[2], φ, as against maximizing returns. We make our assumptions operational by maximizing a weighted sum of average returns and the deviation

[1] Robert Weigand is a professor of finance and business strategy who runs a student investment experience at his university. The student fund managers turn over twice a year, yet the fund performs quite respectably.

[2] This can be interpreted as incorporating the cost of portfolio re-allocation due to brokers or trading fees.

from the equilibrium shares w_i that hold when all assets have the same risk and return, subject to a constraint that enforces risk tolerance. To distinguish the equilibrium shares w_i from the shares s_i, we refer to them as "weights". From the foregoing discussion, the weights are proportional to book value. We emphasize that the optimization is not carried out by any individual or hypothetical planner. Rather, it is a device we use to implement our characterization of the market.

We implement risk tolerance by requiring weighted average risk to equal to overall risk tolerance v^*,

$$\sum_{i=1}^{n} s_i v_i = v^*. \tag{14}$$

The risk tolerance captures investor sentiment, and is expected to change over time. In Keynes' terms, a high value reflects "animal spirits". As we show below, changes in v^* drive changes in money holdings, and thus capture liquidity preference. The change in liquidity preference then affects prices of financial assets, reflecting Minsky's (1980) argument that liquidity preference is linked to the price level of capital assets, and not primarily to interest rates.

The other constraint in the model is the requirement that the shares s_i behave like shares, in that they must sum to one, and are nonnegative,

$$\sum_{i=1}^{n} s_i = 1, \quad s_i \geq 0. \tag{15}$$

4.1 Measuring deviation from the weights

From the argument developed in earlier sections, we expect that investors as a whole will allocate wealth more in less in proportion to the distribution of the book value of underlying assets, deviating only to the extent that risk and return deviates between assets. We implement this by constructing a deviation function $f(x)$ that has a positive second derivative at $x = 0$. Deviation from the weights w_i is measured by a weighted average deviation D,

$$D = \sum_{i=1}^{n} w_i f\left(\frac{s_i}{w_i} - 1\right). \tag{16}$$

By assigning the same deviation function to each asset, we assume that investors apply a criterion of minimizing deviation between market capitalization and book value independent of the underlying asset; only the importance of that asset matters, as measured by its weight.

Different deviation functions reflect different assumptions about behavior. For example, if it is assumed that investors know only the risk and return profile of assets, and use no other information, then we might propose that D measures the information contained in s_i that distinguishes it from w_i. We would then minimize that information content. This corresponds to the Kullback-Leibler (1951) divergence, which can be expressed in the form shown in equation (16). However, as we now show, we do not need to specify the function beyond a single parameter, if we can assume that the shares are close to the weights.

When the shares are close to the weights, we can expand the function f in a Taylor series,

$$f\left(\frac{s_i}{w_i} - 1\right) = f(0) + f'(0)\left(\frac{s_i}{w_i} - 1\right) + \frac{1}{2} f''(0)\left(\frac{s_i}{w_i} - 1\right)^2 + \cdots. \tag{17}$$

Taking the sum from equation (16), we have

$$
\begin{aligned}
D &= f(0)\sum_{i=1}^{n} w_i + f'(0)\sum_{i=1}^{n}\left(s_i - w_i\right) + \frac{1}{2} f''(0)\sum_{i=1}^{n} w_i\left(\frac{s_i}{w_i} - 1\right)^2 + \cdots \\
&= f(0) + \frac{1}{2} f''(0)\sum_{i=1}^{n} w_i\left(\frac{s_i}{w_i} - 1\right)^2 + \cdots.
\end{aligned}
\tag{18}
$$

The first nontrivial term is therefore the quadratic term, and we use the weighted mean squared relative deviation of the shares from the weights. We multiply it by a scale factor r^* to give the term the same units as the mean rate of return,

$$
D \cong r^* \sum_{i=1}^{n} w_i\left(\frac{s_i}{w_i} - 1\right)^2. \tag{19}
$$

The objective function, with Lagrange terms, is a weighted sum of deviations from the weights – representing a desire to minimize turnover – and average return – representing a desire to maximize returns – with the weight on each term determined by the parameter φ,

$$
Z = -\varphi r^* \sum_{i=1}^{n} w_i\left(\frac{s_i}{w_i} - 1\right)^2 + \left(1-\varphi\right)\sum_{i=1}^{n} s_i r_i + \lambda\left(1 - \sum_{i=1}^{n} s_i\right) + \mu\left(v^* - \sum_{i=1}^{n} s_i v_i\right) + \sum_{i=1}^{n} \theta_i\left(a_i^2 - s_i\right). \tag{20}
$$

In (20), the first two components stem from the objective functions and the last three components are due to the three sets of constraints imposed on the shares: (i) the sum of share is equal to 1, (ii) the weighted average of risk is always equal to the targeted risk level of the portfolio, and (iii) a positivity constraint on the shares.

4.2 First-order conditions and solution

Aside from the constraints in equations (14) and (15), the first-order conditions are

$$
\frac{\partial Z}{\partial s_i} = -2r^*\varphi\left(s_i - w_i\right) + \left(1-\varphi\right) w_i r_i - \lambda w_i - \mu w_i v_i - \theta_i w_i = 0, \quad \frac{\partial Z}{\partial a_i} = 2\theta_i a_i = 0. \tag{21}
$$

The final condition is characteristic for a positivity requirement. It is satisfied in one of two cases. First, $a_i = 0$, which means that $s_i = 0$. In that case, θ_i can be nonzero. Second, $\theta_i = 0$, in which case a_i can be nonzero and s_i can take on a positive value. We solve the maximization problem setting all $\theta_i = 0$; that is, we assume a solution exists in which all shares can be nonzero. The case where some θ_i are not zero is relevant in practical applications where some shares may fall to zero, but for our purposes it adds complication without insight. With this assumption, the first-order conditions give the following expression for the shares

$$
s_i = w_i\left(1 + \frac{1-\varphi}{2\varphi}\frac{r_i}{r^*} - \frac{\lambda + \mu v_i}{2r^*\varphi}\right). \tag{22}
$$

Next, we impose the constraints to determine the Lagrange multipliers. First, because shares must sum to one, we have

$$
\left(1-\varphi\right)\sum_{i=1}^{n} w_i r_i - \lambda - \mu\sum_{i=1}^{n} w_i v_i = 0. \tag{23}
$$

Second, the condition that we construct a portfolio with a specified average risk gives

$$-2r^*\varphi\left(v^*-\sum_{i=1}^{n}w_iv_i\right)+(1-\varphi)\sum_{i=1}^{n}w_ir_iv_i-\lambda\sum_{i=1}^{n}w_iv_i-\mu\sum_{i=1}^{n}w_iv_i^2=0. \tag{24}$$

To simplify the notation, use an angle bracket to represent a weighted average with weights given by w_i,

$$\langle x\rangle=\sum_{i=1}^{n}w_ix_i. \tag{25}$$

With this notation, we can write

$$\langle r\rangle=\sum_{i=1}^{n}w_ir_i,\quad \langle rv\rangle=\sum_{i=1}^{n}w_ir_iv_i,\quad \langle v\rangle=\sum_{i=1}^{n}w_iv_i,\quad \langle v^2\rangle=\sum_{i=1}^{n}w_iv_i^2, \tag{26}$$

and we can write a matrix equation for the Lagrange multipliers λ and μ,

$$\begin{pmatrix}1 & \langle v\rangle\\ \langle v\rangle & \langle v^2\rangle\end{pmatrix}\begin{pmatrix}\lambda\\ \mu\end{pmatrix}=\begin{pmatrix}(1-\varphi)\langle r\rangle\\ (1-\varphi)\langle rv\rangle-2r^*\varphi\left(v^*-\langle v\rangle\right)\end{pmatrix}. \tag{27}$$

Solving this equation and substituting into equation (22) for the shares, we find

$$s_i=w_i\left[1+\frac{1-\varphi}{2\varphi}\frac{r_i}{r^*}-\frac{1-\varphi}{2r^*\varphi}\frac{\left(\langle v^2\rangle-v_i\langle v\rangle\right)\langle r\rangle+\left(v_i-\langle v\rangle\right)\langle rv\rangle}{\langle v^2\rangle-\langle v\rangle^2}+\frac{\left(v_i-\langle v\rangle\right)\left(v^*-\langle v\rangle\right)}{\langle v^2\rangle-\langle v\rangle^2}\right]. \tag{28}$$

4.3 Recovering the Tobin equation

Rearranging equation (28), and writing out sums in r_i explicitly, we find that the shares can be written

$$s_i=w_i\left[1+\frac{\left(v_i-\langle v\rangle\right)\left(v^*-\langle v\rangle\right)}{\langle v^2\rangle-\langle v\rangle^2}\right]+w_ia\left[r_i+\sum_{j=1}^{n}w_jr_j\left(\frac{\langle v\rangle v_j-v_iv_j}{\langle v^2\rangle-\langle v\rangle^2}-\frac{\langle v^2\rangle-v_i\langle v\rangle}{\langle v^2\rangle-\langle v\rangle^2}\right)\right], \tag{29}$$

where

$$a=\frac{1-\varphi}{2r^*\varphi}. \tag{30}$$

This is of the right form for the Tobin equation. The first two bracketed term constitutes the vector of base shares,

$$\lambda_{i0}=w_i+w_i\frac{\left(v_i-\langle v\rangle\right)\left(v^*-\langle v\rangle\right)}{\langle v^2\rangle-\langle v\rangle^2}. \tag{31}$$

The λ_{ij} are given by

$$\lambda_{ij}=a\left[\delta_{ij}w_i-w_iw_j\frac{\langle v^2\rangle-\langle v\rangle\left(v_i+v_j\right)+v_iv_j}{\langle v^2\rangle-\langle v\rangle^2}\right]. \tag{32}$$

These coefficients are symmetric, and they satisfy the summing-up condition in equation (3)

$$\sum_{j=1}^{n}\lambda_{ij}=w_ia\left(1-\frac{\langle v^2\rangle-\langle v\rangle v_i+\langle v\rangle^2+v_i\langle v\rangle}{\langle v^2\rangle-\langle v\rangle^2}\right)=0. \tag{33}$$

Also, the base shares sum to one, as in equation (2)

$$\sum_{i=1}^{n} \lambda_{i0} = \sum_{i=1}^{n} w_i + \frac{\left(\langle v\rangle - \langle v\rangle\right)\left(v^* - \langle v\rangle\right)}{\langle v^2\rangle - \langle v\rangle^2} = 1. \tag{34}$$

These are therefore fully acceptable coefficients.

In the limit that all risk measures converge to a common value v, so that assets become indistinguishable aside from their returns, we have

$$\lim_{v_i \to v} \lambda_{i0} \to w_i, \tag{35}$$

$$\lim_{v_i \to v} \lambda_{ij} \to a\left(\delta_{ij} w_i - w_i w_j\right). \tag{36}$$

We therefore recover, in the case of indistinguishable assets, equations (12) and (13).

4.4 Finding an expression for Tobin q*'s*

Using the expressions for the base shares, equation (31), and the λ_{ij}, equation (32), it is possible to construct a set of Tobin coefficients given an assumed equilibrium level of money holding w_n, the distribution of book value between other assets to compute the w_i, the risk associated with each asset by some measure (*e.g.*, idiosyncratic volatility) v_i, the general target level of risk v^*, which expected to change with market conditions, and a parameter a that captures the preference for low turnover as against high return. As a practical matter, this considerably simplifies the use of the Tobin model in simulation models. On a theoretical level, this specification explicitly incorporates risk into the Tobin model, thereby reflecting the common understanding that investors look mainly at the risk-return profile of assets.

We now apply the model by calculating Tobin q's for the different assets. We show that the expressions for the shares can be written in terms of standard measures of mean, dispersion, and correlation, resulting in a compact and interpretable expression for the q's. Starting from equation (28), and using the definition of a given in equation (30), we can write

$$\frac{s_i}{w_i} = 1 + a r_i - a\frac{\left(\langle v^2\rangle - v_i\langle v\rangle\right)\langle r\rangle + \left(v_i - \langle v\rangle\right)\langle rv\rangle}{\langle v^2\rangle - \langle v\rangle^2} + \frac{\left(v_i - \langle v\rangle\right)\left(v^* - \langle v\rangle\right)}{\langle v^2\rangle - \langle v\rangle^2}. \tag{37}$$

Combining terms and rearranging, it is possible to show that this is equivalent to

$$\frac{s_i}{w_i} = 1 + a\frac{\left(\langle v^2\rangle - \langle v\rangle^2\right)\left(r_i - \langle r\rangle\right) + \left(v_i - \langle v\rangle\right)\left(\langle v\rangle\langle r\rangle - \langle rv\rangle\right)}{\langle v^2\rangle - \langle v\rangle^2} + \frac{\left(v_i - \langle v\rangle\right)\left(v^* - \langle v\rangle\right)}{\langle v^2\rangle - \langle v\rangle^2}. \tag{38}$$

This expression contains standard measures of dispersion and correlation. Noting that the (weighted) variance is given by

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2, \tag{39}$$

and the correlation between the r_i and v_i is given by

$$\rho_{rv} = \frac{\langle rv\rangle - \langle v\rangle\langle r\rangle}{\sigma_r \sigma_v}, \tag{40}$$

we can write equation (38) as

$$\frac{s_i}{w_i} = 1 + a\left(r_i - \langle r\rangle\right) + \left(\frac{v^* - \langle v\rangle}{\sigma_v} - a\frac{\sigma_r}{\sigma_v}\rho_{rv}\right)\frac{v_i - \langle v\rangle}{\sigma_v}. \tag{41}$$

This is a suggestive formulation. The final factor is a normalized risk deviation index, which we call z_i,

$$z_i \equiv \frac{v_i - \langle v\rangle}{\sigma_v}. \tag{42}$$

The first term in the parentheses is a measure of how difficult it is to form a portfolio with the desired risk characteristics, given the risk profile of the available assets. Denoting this by R,

$$R \equiv \frac{v^* - \langle v\rangle}{\sigma_v}, \tag{43}$$

we see that R is likely to be close to zero in a “normal” market, in which desired risk is well within one standard deviation of the average. R will be large and positive in a bull market or bubble, in which investors seek investment opportunities among increasingly risky assets. It will be large and negative in a bear market or recession, in which investors view most assets as risky and have a low tolerance for taking on risk.

From equations (8)-(11) for the s_i and w_i, we find equations for the share held as money and for the q’s,

$$\frac{1-m}{1-m^*}\frac{q_i}{\bar{q}} = 1 + a\left(r_i - \langle r\rangle\right) + \left(R - a\frac{\sigma_r}{\sigma_v}\rho_{rv}\right)z_i, \tag{44}$$

$$\frac{m}{m^*} = 1 + a\left(r_n - \langle r\rangle\right) + \left(R - a\frac{\sigma_r}{\sigma_v}\rho_{rv}\right)z_n. \tag{45}$$

In a recession, with R is large and negative, holdings of less risky assets – those with negative values of z_i – go up. This includes money, so there is a flight to liquidity in a down market. Relative values of q are therefore lower for risky assets in a recession. This captures Minsky’s (1980) insight, which he attributes to Keynes, that liquidity preference is related to the price level of capital assets. In our model, risk tolerance v^* is the relevant variable, which can be seen as a measure of “animal spirits”. Changes in v^* drive R positive or negative, thereby driving investors towards or away from holding money, while also changing the price level of capital assets as measured by the q’s.

The relationship between q and risk depends on the correlation between risk and return, ρ_{rv}. From equations (44) and (45), if the correlation is large and positive, then higher risk means lower q, other things remaining the same, while a large negative correlation means that higher risk leads to higher q. It is commonplace to assume that ρ_{rv} is large and positive, because that was the prediction of neoclassical models. However, in a well-known paper, Fama and French (1992) showed that returns were essentially unrelated to risk as measured by asset betas (a correlation-based relative measure). That finding sparked a considerable subsequent literature which has, using various definitions of conditional risk and return, found that the correlation may be large and negative, large and positive, or close to zero. It also spurred researchers to modify their financial models and adopt new definitions of risk. In our extended Tobin model the correlation can be of any sign, and any magnitude. Risk leads investors to shift their asset allocations relative to a risk-free equilibrium as defined by the w_i, but absolute risk has no necessary relationship to absolute return.

5 Conclusion

The Tobin asset allocation model is elegant and useful, and features in stock-flow consistent models. However, it is limited by having a large number of independent variables and determining allocations only on the basis of returns. In this paper we derived a Tobin model in which the coefficients depend on the risk associated with different assets. The resulting formula depends on a small set of parameters that have a sensible economic interpretation: a preference for low turnover as against high returns, and a measure of general risk tolerance. When risk tolerance is high, money flows out of low-risk assets, including money, and into riskier ones, while low risk tolerance leads to a flight to liquidity. The correlation between risk and return appears in the model but, unlike neoclassical financial theory, can take any sign and magnitude.

6 References

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