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Coupling Cycle Mechanisms: Minsky debt cycles and the Multiplier-Accelerator

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Abstract

While there exists a substantial literature on different business cycle mechanisms, there is little literature on economies with more than one business cycle mechanism operating and the relation of stability of these subsystems with the stability of the aggregate system. We construct a model where a multiplier-accelerator subsystem in output-investment space (a real cycle) and a Minskyian subsystem in investment-debt space (a financial cycle) can generate stable/unstable cycles in 2D in isolation. We then derive a theorem showing that if two independent cycle mechanisms that generate stable closed orbits in 2D share a self-destabilizing common variable and the true representation of the system is a fully-coupled 3D system where a weighted average of the common variable is in effect, then the 3D system will generate locally stable closed orbits in 3D if and only if the subsystems have the same frequencies and/or the self-destabilizing effects of the common variable evaluated at the fixed point are equal in both subsystems. Our results indicate that in the presence of multiple cycle mechanisms which share common variables in an economy, the stability of the aggregate economy crucially depends on the frequencies of these sub-cycle mechanisms.

JEL Classification Numbers: C32, E32, E44

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1 Introduction

The modelling of cycle mechanisms has long been a topic of interest in economics, particularly following the Lotka-Volterra formulation of thepopulation dynamics in biological species. While earlier attempts focused on real cycles such as multiplier-accelerator cycles, Metzlerian inventory cycles, Kaldor trade cycle, Rose employment cycles, and Goodwin cycles, since the global financial crisis in 2008, there has been a growing interest in financial cycle mechanisms such as Minskyian debt cycles. However, while there are several papers which have attempted to construct large-scale models where multiple cycle-generating mechanisms operate at the same time, there is a lack of systematic analysis of how their stability properties effect the overall stability of the system.

Sordi & Vercelli (2014) for instance build a 4-D model in successive steps where they obtain a Goodwin cycle and a Minsky cycle operating together. In earlier models, Flaschel & Chiarella (2000, 2006, 2010) have constructed models with Metzlerian inventory cycles, Goodwin cycles and accelerator cycles while Chiarella and Flaschel (2011) presents a demand-driven Goodwin model with inventory cycles and debt dynamics. Similarly, Grasselli and Huu (2018) incorporate Metzlerian inventory sub-cycles developed by Franke (1996) into a model with debt and effective demand. Fazzari et al (2008) present simulation results on a model with a Minsky debt cycle operating via interest rates and an accelerator mechanism for what they consider realistic parameter values but they do not offer a formal analysis of their model. In fact, all these analyses are more interested in generating several cycle mechanisms within a single economy rather than analysing how the behaviour of the aggregate system depends on the characteristics of the sub-cycle mechanisms. This is the precise question which this paper investigates: If there are two business cycle mechanisms with different frequencies operating at the same time in the economy, where these mechanisms share one common variable, how does the aggregate behaviour of the economy depend on the stability properties of these sub-cycle mechanisms? [1]

The issue of different frequencies of cycle mechanisms has been analyzed empirically in Drehmann et al (2012), Borio (2012) and Aikman et al (2013) in the context of real and financial cycles. They show that the cycle in real output displays a smaller frequency (and magnitude) than the financial cycle measured in terms of credit growth and property prices. Thus, the real cycle seem to be shorter and smaller than the financial cycle. Among the few theoretical contributions to modelling multiple cycle mechanisms with different frequencies is Ryoo & Skott (2010) which presents a long financial wave and a short real cycle along Kaldorian lines with Harrodian instability. However, the financial cycle is decoupled from the real cycle in their model and the properties of the emerging long and short cycles in the full system and their relationship with the stability properties of the short and long waves is not analysed in detail. Methodologically, the closest to what we do is Flaschel et al (2005) which constructs a Goodwinian wage share-unemployment cycle in 2D and a Friedmanian inflation-unemployment cycle in 2D and investigates the behaviour of these systems in comparison to a fully-coupled 3D system with inflation, unemployment and wage share. Their analysis however focuses on showing that these two cycle mechanisms

[1] Although papers by Chiarella & Flaschel discuss the interaction of various effects they name as Rose effect, Keynes effect, Mundell effect etc. and the implications of simultaneous operation of these effects on stability of the aggregate economy, their analysis is not on the characteristics of the sub-cycle mechanisms. Therefore, issues such as the effect of different frequencies and magnitudes of sub-cycles on the aggregate system are not investigated systematically.

are compatible with each other as they reproduce the original cycle properties of the 2D systems also in the fully coupled 3D model rather than giving rise to complex oscillations as may be expected in the case of such coupled-oscillators.

We propose a specific model with two distinct cycle mechanisms, namely a two-dimensional multiplier-accelerator model in investment-output space to represent the real cycle and a two dimensional Minsky model in investment-debt space to represent the financial cycle. There is an established literature on multiplier-accelerator models, dating back to the original paper by Samuelson (1939). While the principle of the multiplier was put forward by Kahn (1931) and Keynes (1936), and stated a positive relationship between investment and equilibrium income, the accelerator principle, as argued by Clark (1917), implied that increases in output will also lead to an increase in investment. The feedback between these two principles was analysed by Samuelson (1939) with periodic changes in consumption driving investment, followed by Harrod (1939) and later by Hicks (1950) in the context of coupled difference equations. Goodwin (1951) on the other hand introduced a nonlinear accelerator in continuous time that could generate cycles. Recent contributions in continuous time include Sordi et al (2004), where a multiplier-accelerator cycle around a trend is constructed via a second order differential equation. The literature on Minksy models is more recent, with various attempts since the 1980s to formalise Minsky's rich analysis. Although most models focus on the interrelation between debt accumulation due to investment finance or asset prices as a financial fragility indicator and real variables, so far no canonical Minsky model has emerged. Nikolaidi and Stockhammer (2017) offer a survey of the literature and distinguish between debt cycle models vs asset price models. Within debt-cycle models, they also distinguish between Kaldor (overshooting goods market), Kalecki (stable goods market) or Goodwin (supply-determined) type models. The Minsky model we propose generates cycles in debt and investment with an unstable goods market. It is thus close to what Nikolaidi and Stockhammer (2017) classify as a Kaldor-Minsky model.

We construct our model such that in isolation, these two mechanisms can generate independent stable/unstable oscillations and closed orbits in 2D around their common fixed point and analyse what happens if investment is instead driven by a weighted average of the accelerator effect in the real cycle and profit rate effect in the financial cycle, and the true representation of the economy is a fully-coupled 3D system instead of the 2D subsystems. We then generalise our findings and derive a theorem proving that if two independent cycle mechanisms that generate closed orbits in 2D share a self-destabilizing common variable and the true representation of the system is a fully-coupled 3D system where a weighted average on the common variable is in effect, then the 3D system will generate locally stable closed orbits if and only if the subsystems have the same frequencies and/or the self-destabilizing effects of the common variable evaluated at the fixed point are equal in both subsystems. In the context of our multiplier-accelerator and Minskyian sub-systems, the theorem also implies that if the subsystem with a stronger (weaker) effect on investment also has a lower frequency, the operation of both cycle mechanisms has a stabilizing (destabilizing) effect on the economy and the combination of two closed orbit systems lead to a fully-coupled 3D system that produces dampening oscillations.

As a numerical example, we parameterise the model such that the two dimensional sub-systems generate locally asymptotically stable closed orbits. In line with stylized facts the accelerator effect on investment is larger than the Minskyian profit rate effect and the multiplier-accelerator cycle has a higher frequency. Through numerical simulations, we show that in line with our analytical results, there is an inherent stabilizing effect in this case and the economy displays dampening cycles around the fixed point if the multiplier-accelerator and Minsky mechanisms operate with a weighted effect as described above. We note that our analytical results also imply that with longer financial cycles than real cycles and a stronger profit rate effect on investment than the accelerator effect, the aggregate economy may also be unstable and display explosive cycles although individual real and financial sub-cycles display stable oscillations.

The paper is organized as follows. Section 2 presents the description and solution of the model. Section 3 derives a general theorem for coupling two cycle mechanisms while Section 4 presents a numerical demonstration of the analytical results derived in the theorem via simulations. In Section 5 we provide sensitivity analysis of the numerical simulations. The last section concludes with some remarks and blueprints for future research.

2 The Model

We assume a closed economy which produces a single good. Output is produced using labour and capital in fixed proportions and it is given by Y. Per capita output is defined as output capital ratio and denoted as $y = Y/K$.

We assume that output follows an excess demand adjustment process in per capita terms as in Asada (2001)[2]

$$\dot{y} = \alpha(c + g - y), \quad \alpha > 0 \tag{1}$$

where $g = I/K$ is the growth rate of investment and $c = C/K =$ per capita consumption.

Aggregate consumption can be written as:

$$C = C_w + C_c \tag{2}$$

where as in the Kaleckian literature, workers consume all wages:

$$C_w = Wage\ Bill \tag{3}$$

With pricing taking the form of a constant mark-up over average wage costs and constant labour productivity, consumption out of wage income becomes

$$C_w = (1 - \pi)Y \tag{4}$$

where $(1 - \pi)$ is the constant wage share. It must however be noted that the π in this case includes interest payments $i.d$ as well as firm profits and therefore assuming a uniform average propensity to consume out of profits and interest income, we can write equation (5) below as a consumption function for all non-wage income.

$$C_c = m_c \Pi^e \tag{5}$$

[2]In essence, one other possible way is to assume that output adjustment takes place according to excess demand in gross quantities as $\dot{Y} = \alpha(C + I - G)$ and derive the dynamics of per capita output adjustment using $\dot{y} = \dot{Y}/K - (Y/K)(\dot{K}/K)$. However in this case, setting $\dot{y} = 0$ in the steady state will imply that output and capital must be growing at the same rate and goods market can be in disequilibrium in the long run. The current formulation on the other hand implies that setting $\dot{y} = 0$ ensures goods market equilibrium where $C + I = Y$.

where m_c is the uniform average propensity to consume[3].

Dividing (2) and (3) by K , we get

$$c = (1-\pi)y + m_c r^e \tag{6}$$

where expected profit rate is given by

$$r^e = \pi y^e \tag{7}$$

As in Gandolfo (1978) and in Sordi et al (2014), we assume extrapolative expectations[4]:

$$y^e = y + \tau \dot{y} \tag{8}$$

which implies

$$r^e = \pi(y + \tau \dot{y})$$

Plugging this equation into (6)

$$c = (1-\pi)y + m_c \pi(y + \tau \dot{y}) \tag{9}$$

We can now derive the dynamics of per capita output. From (1) and (9),

$$\dot{y} = \alpha \left[(1-\pi)y + m_c \pi(y + \tau \dot{y}) + g - y\right]$$

Simplifying and defining $\theta = \alpha \tau m_c \pi$, we get

$$\dot{y}(1-\theta) = \alpha \left[(m_c - 1)\pi y + g\right]$$

Assuming that goods market does not adjust too rapidly and expectations are not very strongly extrapolative (α and τ not very high), $\theta < 1$ will always hold.

Denoting $\dfrac{\alpha}{(1-\theta)} = \Omega > 0$ and $m_c - 1 = -s_c$ where s_c is the marginal propensity to save out of expected profits, the dynamics of output-capital ratio boils down to:

$$\dot{y} = \Omega \left[g - s_c \pi y\right] \tag{10}$$

We assume investment dynamics are determined by an adjustment process depending on the deviation of desired growth rate of investment from the actual rate, as in Jarsulic (1990, 1996), Charles (2008).

$$\dot{g} = \epsilon \left[g^d - g\right], \quad \epsilon > 0 \tag{11}$$

[3]Here, we implicitly assume that profit-earners have stocks of wealth out of which they can consume should the actual distributed profits fall below the level they decide to consume.

[4]Since $y = Y/K = (Y/Y_{fc})(Y_{fc}/K)$ where Y_{fc} is full-capacity output, assuming constant capital productivity and therefore constant Y_{fc}/K implies that output-capital ratio, y, can be considered as a measure of capacity utilization. Therefore, our specification is identical to assuming extrapolative expectations on capacity utilization rather than gross output (Y).

The desired investment function consists of two parts: A multiplier-accelerator mechanism which links growth rate of investment to output growth and a Minskyian part that takes into account the effect of debt on investment

$$g^d = g_0 + \beta [v\dot{y}] + (1-\beta)[\gamma r - \lambda d], \tag{12}$$
$$MA \qquad MINSKY \tag{13}$$

where $d = D/K =$ per capita stock of debt, r is the aggregate profit rate on capital and λ is a parameter that measures the impact of higher debt on the willingness of firms to borrow more, as in Charles (2015)[5]. A higher debt level implies that firms are less enthusisatic to increase the stock of their debt (or banks are less enthusiastic to lend to firms).

In line with the pecking order of finance, firms borrow to finance investment in excess of retained profits net of interest payments. As argued by Passarella (2012: 574) as well, from a macroeconomic perspective, leverage can increase during a boom only if interest rates increase or retention rate falls. We assume constant interest rates but an endogenous retention rate that depends on the actual profit rate negatively. As also documented by Charles (2008), Benartzi, Michaely and Thaler (1997) find that firms cut dividends as their profits fall and our specification captures this observation in a simple way. The dynamics of debt accumulation is therefore given by

$$\dot{D} = I - (1-\phi r)(\Pi - iD) \tag{14}$$

where $s_f = (1-\phi r)$.

2.1 A Pure Multiplier-Accelerator System ($\beta = 1$)

When $\beta = 1$ in (12), investment is driven purely by the multiplier-accelerator term and takes the form:

$$\dot{g} = \epsilon [g_0 + v\dot{y} - g] \tag{15}$$

Substituting (10) gives

$$\dot{g} = \epsilon g_0 + \epsilon(\Omega v - 1)g - \epsilon \Omega s_c \pi v y \tag{16}$$

Using $\dot{d} = \dot{D}/K - (D/K)(\dot{K}/K)$ and assuming zero depreciation so that $I = \dot{K}$, and therefore $\dot{K}/K = g$, we find the dynamic equation for debt-capital ratio as

$$\dot{d} = g - (1-\phi r)(r - id) - dg \tag{17}$$

In order to be able to derive the dynamics of the debt-capital ratio, we need to derive the actual profit rate in the economy. Gross profits are given by[6]

[5] Charles (2008) uses $g^d = g_0 + \gamma s_F(r - id)$ while Charles (2015) has an investment function of the form $I = I_0 + \beta s_F(\Pi - iD) - \lambda D$.

[6] Note that with a goods market disequilibrium as in (1), firms must be holding inventories so that no rationing occurs in the case of excess demand and the profit share π as defined in this model therefore includes the accumulation of inventories in the case of excess supply in the goods market. Therefore, gross profit rate r, which is the valid variable for debt accumulation, is *not* given by $r = \pi y$, as that would mean unsold goods piling up in inventories are being used to pay back debt. On the other hand, expected profit rate can be defined as $r^e = \pi y^e$ as in (7) unless firms expect excess supply in the goods market or produce in order to attain a target inventory level.

$$\Pi = Sales - Costs$$

Assuming that as well as wages, firms also have other costs of production proportionate to production level, such as imported energy costs, given by μY, profits can be written as

$$\Pi = C + I - Wage\ Bill - \mu Y$$

Using (3) and (5),

$$\Pi = m_c \Pi^e + I - \mu Y$$

Dividing by K,

$$r = \frac{\Pi}{K} = m_c \pi (y + \tau \dot{y}) + g - \mu y$$

Substituting (10) and simplifying, we get

$$r = \Sigma g + \Gamma y \tag{18}$$

where $1 + m_c \pi \Omega \tau = \Sigma > 1$ and $m_c \pi (1 - \Omega \tau s_c \pi) - \mu = \Gamma$

Assumption 1: $\Gamma = m_c \pi (1 - \Omega \tau s_c \pi) - \mu = 0$[7].

Using (18), equation (17) can be written as:

$$\dot{d} = g(1-d) - (1 - \phi \Sigma g)(\Sigma g - id) \tag{19}$$

Together with the equations of motion for y and g derived above and repeated below, we have a three-dimensional non-linear system.

$$\dot{y} = \Omega \left[g - s_c \pi y \right] \tag{20}$$

$$\dot{g} = \epsilon g_0 + \epsilon (\Omega v - 1) g - \epsilon \Omega s_c \pi v y \tag{21}$$

2.1.1 Steady State

Setting the equations above to zero simultenously and solving, we can derive the fixed points of the system.

Proposition 1 *The system (19) - (21) has the fixed point* $(y_A, g_A, d_A) = (g_0 / s_c \pi, g_0, \frac{\sigma_1 g_0 [g_0 - \sigma_2]}{g_0 - \sigma_3})$ *where* $\sigma_1 = \frac{\phi \Sigma^2}{1 + \phi \Sigma i}, \sigma_2 = \frac{\Sigma - 1}{\phi \Sigma^2}$ *and* $\sigma_3 = \frac{i}{1 + \phi \Sigma i}$.

Proof. See Appendix A1. ■

Assumption 2: $g_0 > \sigma_3$ & $g_0 > \sigma_2$

Under this assumption, the system has a unique non-trivial fixed point with a positive value for the debt-capital stock ratio.

[7] As we will discuss again, this assumption is only to ensure that in the pure Minsky system we will analyze in the next part, investment dynamics do not depend directly on y so that we can have an independent system in g and d.

2.1.2 Stability

In order to analyze the local stability properties of the multiplier-accelerator model, let us derive its Jacobian matrix.

$$\mathbf{J}^A = \begin{bmatrix} -\Omega s_c \pi & \Omega & 0 \\ -\epsilon \Omega s_c \pi v & \epsilon(\Omega v - 1) & 0 \\ 0 & 1 + 2\phi \Sigma^2 - d_A^* - \Sigma - \phi \Sigma i d_A^* & -(1+\phi\Sigma i) g_0 + i \end{bmatrix},$$

where d_A^* and g_0 are the fixed points of the multiplier-accelerator system. The eigenvalues of this system can be found by finding the eigenvalues of the multiplier-accelerator subsystem given by $(3,3)$ minor of $\mathbf{J}^A$:

$$\mathbf{J}_S^A = \begin{bmatrix} -\Omega s_c \pi & \Omega \\ -\epsilon \Omega s_c \pi v & \epsilon(\Omega v - 1) \end{bmatrix}$$

and the third eigenvalue is equal to $-(1+\phi\Sigma i) g_0 + i$. Using the Routh-Hurwitz conditions, we can establish the following proposition.

Proposition 2 *The system (19) - (21) is stable when* $v < v^* = \dfrac{\epsilon + \Omega s_c \pi}{\epsilon \Omega}$, *it unstable for* $v > v^*$. *At* $v = v^*$, *the system goes through a Hopf bifurcation and locally asymptotically stable closed orbits emerge.*

Proof. See Appendix B1. ■

As shown in Appendix B1 , the bifurcation parameter v (sensitivity of investment to the change in output) does not affect the fixed point of the system.

2.2 A Pure Minsky System ($\beta = 0$)

When $\beta = 0$, the multiplier-accelerator part of the investment function vanishes and investment is given by only Minskyian dynamics. Using (11) (12) and (18) with Assumption 1, we have a pure Minskyian system which is defined by

$$\dot{y} = \Omega \left[g - s_c \pi y \right] \tag{22}$$

$$\dot{g} = \epsilon \left[g_0 + (\gamma \Sigma - 1) g - \lambda d \right] \tag{23}$$

$$\dot{d} = g(1-d) - (1 - \phi \Sigma g)(\Sigma g - i d) \tag{24}$$

This time, the growth rate of investment and debt-capital ratio can be solved together as a dynamic system, which we will call Minskyian subsystem.

2.2.1 Steady State

Unlike the multiplier-accelerator model, the fixed points of the system (22) - (24) cannot easily be solved for analytically but a graphical representation is fairly straightforward. Setting $\dot{g} = 0$ and $\dot{d} = 0$, we get respectively

$$d = g_0/\lambda + (\gamma\Sigma - 1)/\lambda g = h(g) \tag{25}$$

$$d = \frac{\sigma_1 g\,[g - \sigma_2]}{g - \sigma_3} = f(g) \tag{26}$$

As shown in Appendix A2, the number of the fixed points and their stability properties depend on the relative sizes of σ_1, σ_2, σ_3 and the sign of $(\gamma\Sigma - 1)$. We can summarize some of the results as below:

Proposition 3 *If $\phi < \frac{\Sigma - 1}{\Sigma i}$ holds, then $\sigma_2 > \sigma_3$ and nullcline for the debt-capital ratio takes the shape in Figure 1 and 2 below. In this case, $\sigma_1 > (\gamma\Sigma - 1)/\lambda > 0$ implies that there are two equilibria as shown in Fig 1. If $0 < \sigma_1 < (\gamma\Sigma - 1)/\lambda$ on the other hand, the system has a single fixed point (Fig. 2).*

Proof. See Appendix A.1 and A.2 for a complete characterization of the phase diagrams under certain parameter configurations ■

2.2.2 Stability

The Jacobian of the Minsky system is given by:

$$\mathbf{J}^M = \begin{bmatrix} -\Omega s_c \pi & \Omega & 0 \\ 0 & \epsilon(\gamma\Sigma - 1) & -\epsilon\lambda \\ 0 & 1 + 2\phi\Sigma^2 g_M^* - d_M^* - \Sigma - \phi\Sigma i d_M^* & -(1 + \phi\Sigma i) g_M^* + i \end{bmatrix} \tag{27}$$

where d_M^* and g_M^* are the fixed points.

As above, the eigenvalues of the system can be calculated by finding the eigenvalues of the Minskyian subsystem given by $(1, 1)$ minor of $\mathbf{J}^M$, which we call $\mathbf{J}_S^M$ and the third eigenvalue is equal to $-\Omega s_c \pi < 0$.

$$\mathbf{J}_S^M = \begin{bmatrix} \epsilon(\gamma\Sigma - 1) & -\epsilon\lambda \\ 1 + 2\phi\Sigma^2 g - d_M^* - \Sigma - \phi\Sigma i d_M^* & -(1 + \phi\Sigma i) g_M^* + i \end{bmatrix} \tag{28}$$

The trace of this matrix is given by

$$Tr(\mathbf{J}_S^M) = \epsilon(\gamma\Sigma - 1) - (1 + \phi\Sigma i) g_M^* + i$$

Figure 1: Phase Diagram for Minsky Subsystem

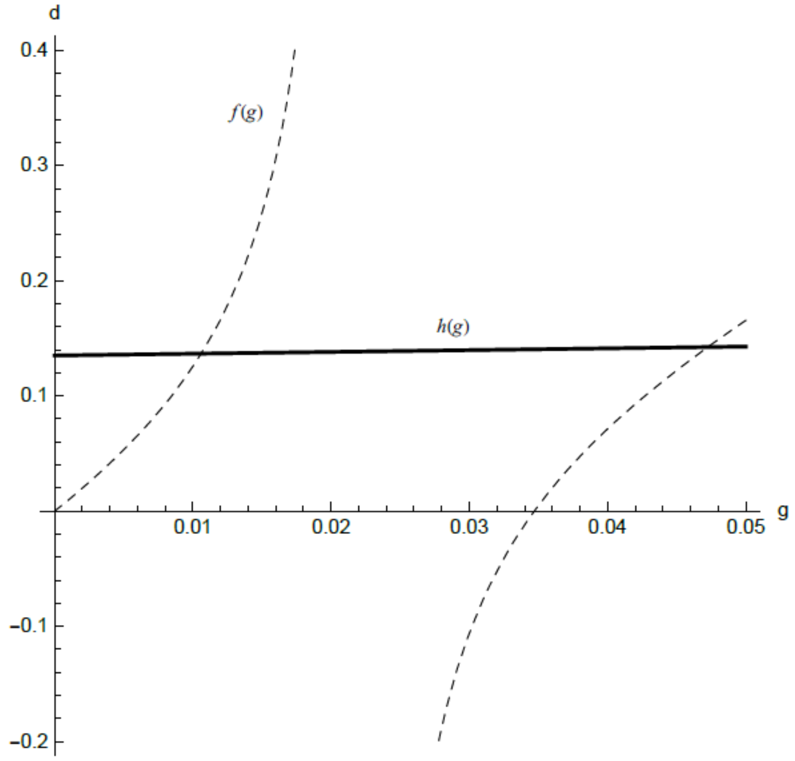
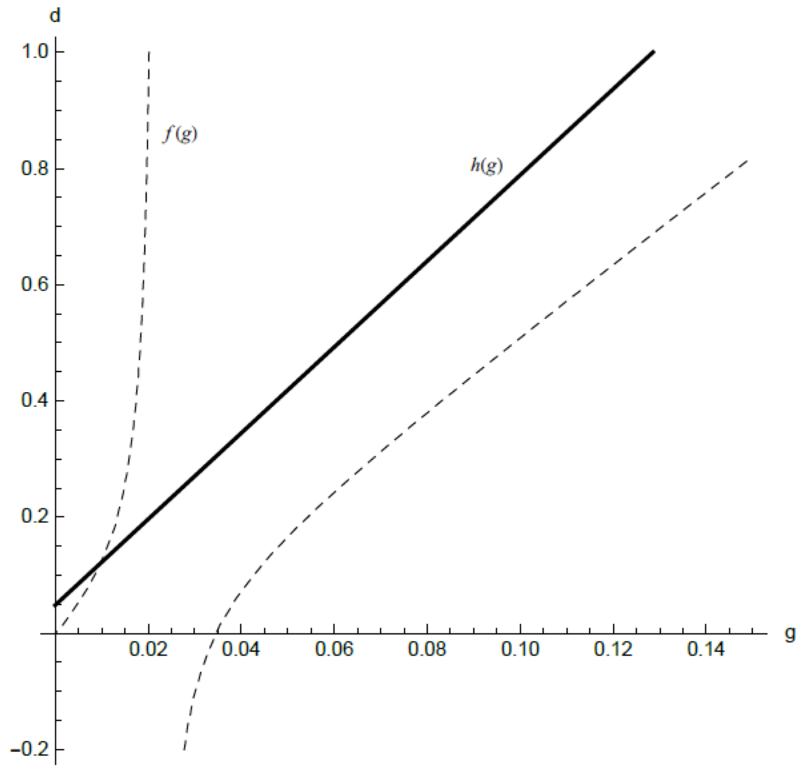


Figure 2: Phase Diagram for the Minsky Subsystem



Since we assumed $g_0 > \sigma_3$ above, $-(1+\phi\Sigma i)g_0 + i < 0$ holds by assumption. This implies that if we want to have a common fixed point $g^* = g_0 = g_M$ for both models, we will also have $-(1+\phi\Sigma i)g_M^* + i < 0$. Therefore the sign and magnitude of $\epsilon(\gamma\Sigma - 1)$ will determine the sign of $Tr(\mathbf{J}_S^M)$ and in order to be able to generate unstable behaviour and closed orbits for the Minsky model, we will assume $\gamma\Sigma > 1$. This gives a Minskyian model where $d(\dot{g})/dg > 0$ and $d(\dot{d})/dd < 0$. In the jargon of Nikoliadi & Stockhammer (2017), we thus have a Kaldorian Minsky model.

The stability of the fixed points can be analyzed graphically by deriving the following proposition:

Proposition 4 *Assume that $\phi < \frac{\Sigma-1}{\Sigma i}$ holds so $\sigma_2 > \sigma_3$. If $h(g)$ line cuts the $f(g)$ curve from above at a point below the asymptote at σ_3, the determinant of J_S^M above is negative and the fixed point is a saddle. If $h(g)$ line cuts the $f(g)$ curve from above at a point above σ_3, $Det(J_S^M) > 0$ holds, if it cuts the $f(g)$ curve from below at a point above σ_3, $Det(J_S^M) < 0$ holds and the fixed point is a saddle again.*

Proof. See Appendix B2. ■

Therefore, in Figure 1 above, the first fixed point is always a saddle while the stability of the second fixed point will depend on the sign of $Tr(\mathbf{J}_S^M)$. Setting $Tr(\mathbf{J}_S^M) = 0$ in (29) under the assumption $\gamma\Sigma > 1$, we can establish the following proposition:

Proposition 5 *For every high growth fixed point of the Minsky system $g_M^* > \sigma_3$, there exists an $\epsilon^* = \frac{(1+\phi\Sigma i)g_M^* - i}{(\gamma\Sigma - 1)} > 0$ at which the Minksyian subsystem undergoes a Hopf bifurcation and locally asymptotically stable closed orbits emerge.*

Proof. See Appendix B2. ■

2.3 Combined System ($0 < \beta < 1$)

Next, we can consider a system where both investment dynamics operate at the same time and are weighted by β. In such a case, the differential equation system becomes

$$\dot{y} = \Omega\left[g - s_c\pi y\right]$$

$$\begin{aligned} \dot{g} &= \epsilon\left\{g_0 + \beta v\Omega(g - s_c\pi y) + (1-\beta)(\gamma\Sigma g - \lambda d) - g\right\} \\ \dot{d} &= g(1-d) - (1-\phi\Sigma g)(\Sigma g - id) \end{aligned} \tag{30}$$

2.3.1 Steady State

Since we will combine the two models in this section, we need to ensure that the two separete systems yield (at least one) common fixed point. To do so, ensuring $g_M^* = g_0$ will suffice as output-capital ratio and debt-capital ratio dynamics are identical in both models. From (23), a sufficient condition for this is $\gamma\Sigma g = \lambda d$, which implies that $d = (\gamma\Sigma/\lambda)g$. Denoting $\gamma\Sigma/\lambda = \kappa$, substituting this result in (24) and setting it to zero, we get

$$g(1 - \kappa g) - (1 - \phi\Sigma g)(\Sigma g - i\kappa g) = 0$$

Simplifying yields

$$g^2(\phi\Sigma^2 - \kappa - \phi\Sigma i\kappa) - g(\Sigma - 1 - i\kappa) = 0.$$

The roots of this equation are $g = 0$ and

$$g = \frac{\Sigma - 1 - i\kappa}{\phi\Sigma^2 - \kappa - \phi\Sigma i\kappa}$$

Setting g_0 equal to this value ensures that both multiplier-accelerator and Minsky systems have one common fixed point.

Assumption 3: $g_0 = \dfrac{\Sigma - 1 - i\kappa}{\phi\Sigma^2 - \kappa - \phi\Sigma i\kappa}. = g_M^* = g_A^*$

Intuitively, g_0 gives the trend rate of growth around which fluctuations occur in both models and it is set by the parameters of the model. In order to clarify this assumption mathematically, note that as shown above, setting the growth of investment to zero in the Minskyian subsystem gives $d = g_0/\lambda + [(\gamma\Sigma - 1)/\lambda]\, g = h(g)$. This is the straight line in Figure 1 and Figure 2 above. The intercept of this line is g_0/λ and its slope is $(\gamma\Sigma - 1)/\lambda$. The point where this line intersects the curve to the right of the asymptote gives the non-saddle fixed point of the Minsky system, as derived in Proposition 4. Since the asymptotes in Figures 1 and 2 occur at $g = \sigma_3$[8] and we have assumed in Assumption 2 that $g_0 > \sigma_3$,the non-saddle fixed point of the Minsky system and the fixed point of the multiplier-accelerator system will be equal to g_0 as long as Assumption 3 holds.

Note that since $g = g_0$ is a solution to both (15) and (23), it will also be a solution to (30), which is a linear combination of these two equations. Therefore, Assumption 2 and Assumption 3 also ensure that the combined system will have g_0 as one of its fixed points.

2.3.2 Stability

The stability of the system will now depend on the eigenvalues of the new Jacobian:

$$\mathbf{J} = \begin{bmatrix} -\Omega s_c \pi & \Omega & 0 \\ -\beta\epsilon\Omega v s_c \pi & \epsilon\beta\Omega v + (1-\beta)\epsilon\gamma\Sigma - \epsilon & -\epsilon\lambda \\ 0 & 1 + 2\phi\Sigma^2 g^* - d^* - \Sigma - \phi\Sigma i d^* & -(1+\phi\Sigma i)g^* + i \end{bmatrix}$$

[8]See Appendix A.1

where $g^* = g_0$ and $d^* = \kappa g_0$ are the common fixed points of both subsystems. The Routh-Hurwitz conditions require that

$$a_1 = Tr(\mathbf{J}) < 0,\ a_2 = Det(\mathbf{J}) < 0,\ a_3 = |\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| > 0,$$

and

$$-a_1 a_3 + a_2 > 0,$$

where

$$Tr(\mathbf{J}) = -\Omega s_c \pi + \beta\epsilon\Omega v + (1-\beta)\epsilon\gamma\Sigma - \epsilon - (1+\phi\Sigma i)g_c^* + s_F i \lesseqgtr 0 \tag{31}$$

$$|\mathbf{J}_1| = Det \begin{bmatrix} \epsilon\beta\Omega v + (1-\beta)\epsilon\gamma\Sigma - \epsilon & -\epsilon\lambda \\ 1 + 2\phi\Sigma^2 g^* - d^* - \Sigma - \phi\Sigma i d^* & -(1+\phi\Sigma i)g^* + i \end{bmatrix} \lesseqgtr 0$$

$$|\mathbf{J}_2| = Det \begin{bmatrix} -\Omega s_c \pi & 0 \\ 0 & -(1+\phi\Sigma i)g^* + i \end{bmatrix} > 0$$

and

$$|\mathbf{J}_3| = Det \begin{bmatrix} -\Omega s_c \pi & \Omega \\ -\beta\Omega\epsilon v s_c \pi & \epsilon\beta\Omega v + (1-\beta)\epsilon\gamma\Sigma - \epsilon \end{bmatrix} \lesseqgtr 0$$

3 A Generalization

The stability properties of the 3D system can be generalized by the theorem below[9].

Theorem 1 *Consider any two subsytems S_1 and S_2 given by*

$$S_1 = \begin{array}{l} \dot{x} = f(x,y) \\ \dot{y} = h(x,y) \end{array}$$

and

$$S_2 = \begin{array}{l} \dot{y} = p(y,z) \\ \dot{z} = q(y,z) \end{array}$$

where f, h, p and q are C^1 and have a common fixed point y^*, with $h_y > 0$, $p_y > 0$ so the common variable y has a destabilizing effect on itself in both subsystems. If these mechanisms are in fact operating together with a weighted effect on y such that the true representation of the dynamics is given by the 3D system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \beta h(x,y) + (1-\beta)p(y,z) \\ \dot{z} &= q(y,z) \end{aligned}$$

[9] For proof, see Appendix B.3

then,

a) If the common fixed point is a saddle point in both subsystems, it is also a saddle point in the fully-coupled system.

b) If S_1 and S_2 generate locally asymptotically stable closed orbits in $x - y$ and $y - z$ spaces around y^*, then the 3D system will generate locally asymptotically stable closed orbits around y^* if and only if $h_y = p_y$ or the subsystems have the same frequency. If none of these conditions hold, then the 3D system will be stable if the subsystem with a larger destabilizing effect of the common variable on itself also has a higher frequency, i.e repeats itself more often; it will be unstable if the subsystem with a larger destabilizing effect the common variable on itself has a lower frequency.

c) There is an in-built stabilizing/destabilizing effect if S_1 and S_2 operate in a fully-coupled system as defined above, which depends on the sign of $(h_y - p_y)(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|)$. If $(h_y - p_y)(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|) > 0$, coupling the subsystems has a stabilizing effect and the fully-coupled system is more stable than individual subsystems. If $(h_y - p_y)(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|) < 0$ on the other hand, the fully-coupled 3D system is more unstable than the individual systems.

d) Any parameter change which destabilizes S_1 or S_2 subsystems also destabilizes the fully-coupled system.

e) If on the other hand, $h_y < 0$, and $p_y < 0$ hold so the common variable has a stabilizing effect on itself, then the combination of two closed orbit subsystems will always yield a saddle point for the 3D system. Or in other words, the weighted average of a self-destabilizing process should be offset by two self-stabilizing processes for the stability of the 3D system to be a possibility.

Therefore, if the fixed point is a saddle point in both multiplier-accelerator and Minksy subsystems, it is also a saddle point in the combined system. As a corollary of (c) above, the combination of two stable subsystems may yield an unstable combined system, especially if their stability is only marginal, while two marginally unstable subsystems may yield a stable fully-coupled system and possible bifurcations depending on the relative weights of the two mechanisms (i.e the size of β). Similarly, the combination of stable and unstable subsystems will always lead to ambigous results and the dynamic behaviour/stability of the combined system depends on parameter configurations as well as the size of β. Further, as the self-destabilizing effects of the common variable in each subsystem or the determinants of the subsystems get closer to each other, the inherent stabilising or destabilizing effect of full-coupling, given by $(h_y - p_y)(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|)$, falls.

Corollary 1 *If a multiplier-accelerator and a Minsky subsystem, which generate locally asymptotically stable closed orbits around a common fixed point, are allowed to operate at the same time in the economy as defined in Theorem 1, the combined system will also generate locally asymptotically stable closed orbits around this common fixed point if and only if $\alpha v = \gamma$ and/or $\left|J_S^A\right| = \left|J_S^M\right|$ holds (i.e the accelerator effect on investment is equal to the profit rate effect on investment and/or both subsystems have the same periodicity). If $\alpha v > \gamma$ so that the accelerator effect on investment is larger than profit rate effect on investment, then the combined model will be stable if the accelerator subsystem also has a higher frequency ($\left|J_S^A\right| > \left|J_S^M\right|$); it will be unstable if the accelerator system has a lower frequency ($\left|J_S^A\right| < \left|J_S^M\right|$).*

4 Numerical Simulations

We now calibrate our model in order to match some of the stylized facts documented by Borio (2012) and and Aikman et al (2013). As both studies find, the real cycle has a higher frequency than the financial cycle. Interpreting our multiplier-accelerator model as the real cycle and our Minsky model as the financial cycle, we generate these two cycle mechanisms with this characteristic.[10]

The table below shows the initial values of the parameters. The values for the interest rate, profit share and marginal propensity to save out of profits are standard in calibration of small-scale Minskyian models. We assume a strong dependence of the retention rate on the profit rate (ϕ) and strongly extrapolative expectations (τ) to incorporate the large effect of consumption on profits during a financial-Minksy cycle. We provide sensitivity analysis with respect to the other parameters in Section 4.

Parameter	Definition	Value
i	Interest rate	0.025
λ	Sensitivity of Investment to debt	0.5
ϕ	Sensitivity of retention rate to profit rate	4.5
π	Profit Share	0.4
s_c	Marginal propensity to save out of profits	0.75
γ	Sensitivity of investment to profit rate	0.85
α	Speed of goods market adjustment	0.6
τ	Extrapolativeness of expectations	3.2

Using these, we get the following values for the composite parameters:

Parameter	Value
σ_1	6.05
σ_2	0.034
σ_3	0.022
Ω	0.74
Σ	1.237

Inserting these composite parameters into Assumption 3 gives $g_0 = 0.041$ as the trend rate of growth and the common fixed point for both subsystems becomes $g^* = 0.041$, $y^* = 0.137, d^* = 0.0865$. Therefore, $g^* > \sigma_2 > \sigma_3$ holds. Since $\gamma\Sigma > 1$ and $\sigma_1 > (\gamma\Sigma - 1)/\lambda$ with these parameters, the phase diagram of the Minsky subsystem in g and d (i.e when $\beta = 0$), has the shape in Figure 1 and as we conjectured in Proposition 3, (y^*, g^*, d^*) is the non-saddle fixed point. The Minsky system has also a saddle fixed point at $g^{**} = 0.007$, $y^{**} = 0.024$ and $d^{**} = 0.083$.

[10] In essence, both studies show that the real cycle is also smaller in magnitude. In models of differential equation systems with closed orbits, the magnitude of these orbits depend on the initial values. We do not therefore focus on this issue explicitly. The codes for all simulations are available upon request.

Figure 3:3D Accelerator Cycle

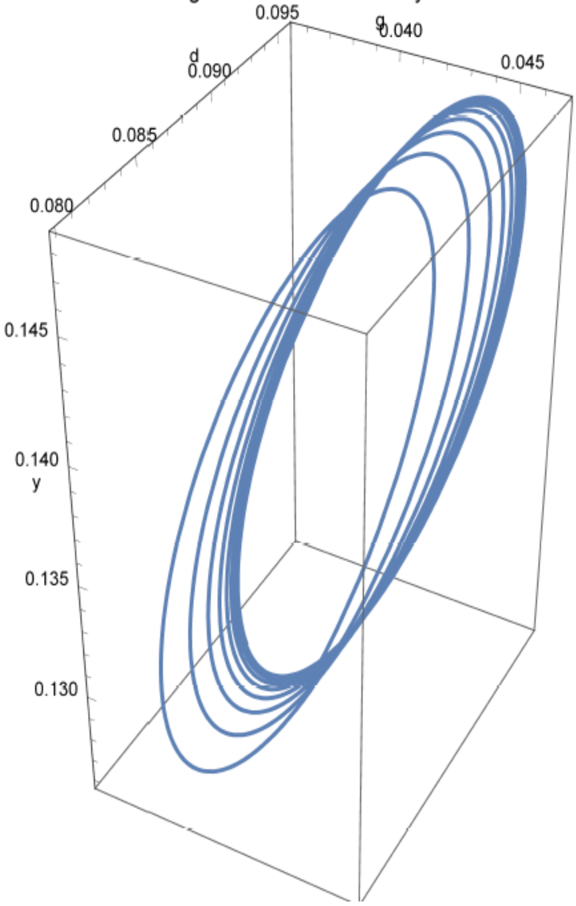


Figure 4:2D Accelerator Cycle in y–g

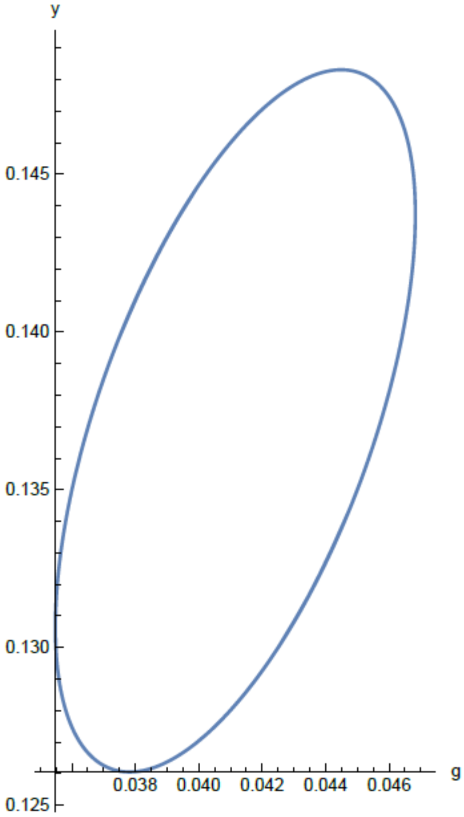


Figure 5:3D Minsky Cycle

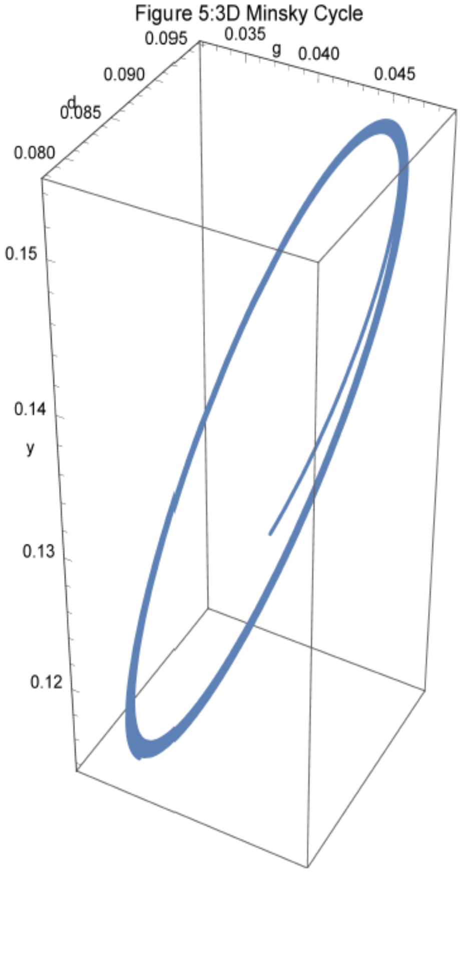


Figure 6:2D Minsky Cycle in g–d

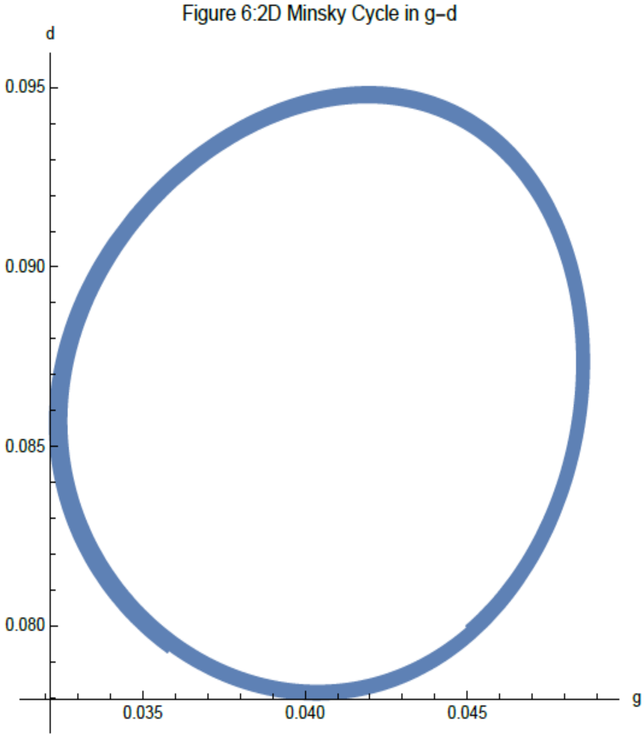


Figure 7: Tr[J]<0

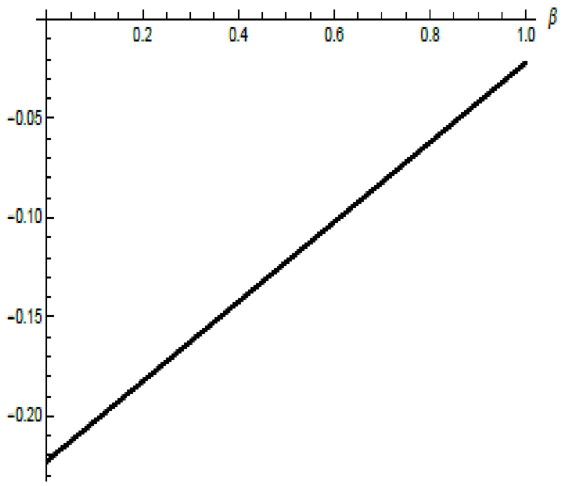


Figure 8: Det[J]<0

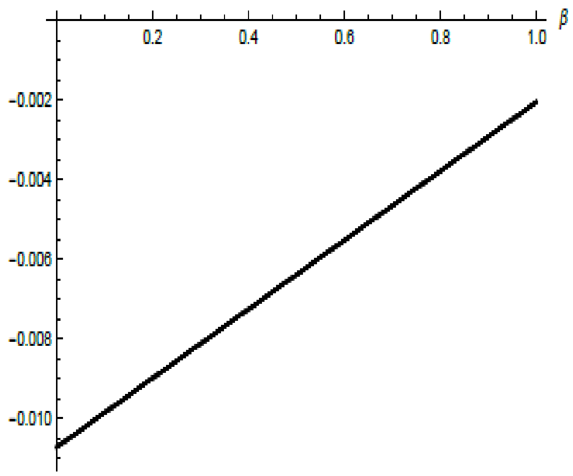
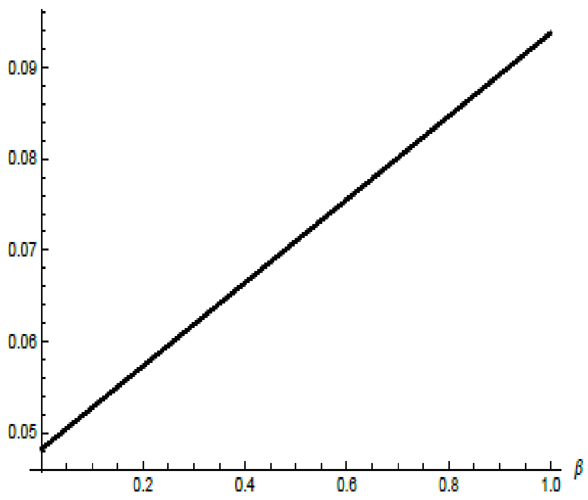


Figure 9: PM[J]>0



-Figure 10: -Tr[J]PM[J]+Det[J]>0

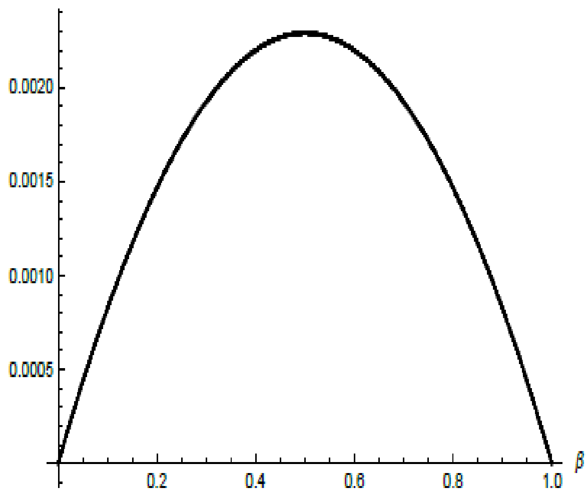
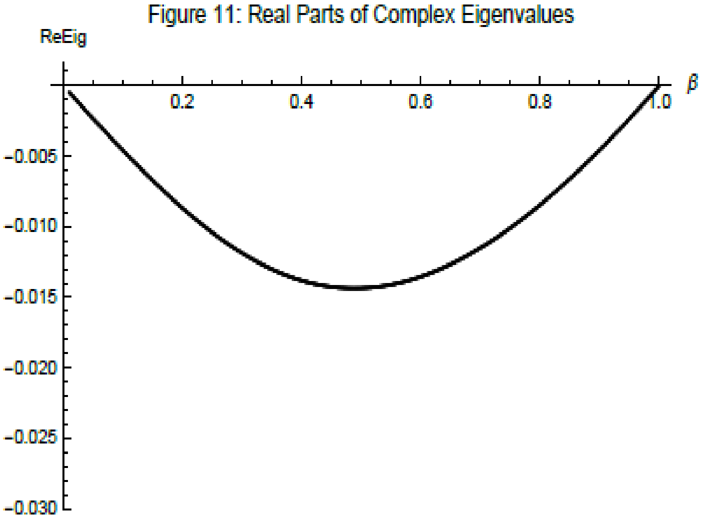


Figure 11: Real Parts of Complex Eigenvalues



Setting $\beta = 1$ and using Proposition 2, we find that at the critical value $v^* = 2.059$, the multiplier-accelerator model generates locally asymptotically stable closed orbits around (y^*, g^*, d^*). Similarly, setting $\beta = 1$ and making use of Proposition 5 with $v = v^*$ reveals that the Minsky system also generates asymptotically stable closed orbits around (y^*, g^*, d^*) when $\epsilon^* = 0.421$. Below, we depict these locally asymptotically stable closed orbits for both systems. As the sign structure of the Jacobians of the subsystems show, growth rate of investment is prey to output-capital ratio in the multiplier-accelerator subsystem while it is prey to debt-capital ratio in the Minsky subsystem. As a result, the subsystem dynamics in the simulations reproduce this relationship where investment exhibits counter-clockwise cycles with output-capital ratio in Figure 4 and counter-clockwise cycles with debt-capital ratio in Figure 6[11].

Next we move on to allowing two cycle generating mechanisms operate at the same time ($0 < \beta < 1$). As we noted in the theorem above, this has an in-built stabilising or destabilising effect, depending on the sign of $(\alpha v - \gamma)(\left|\mathbf{J}_S^A\right| - \left|\mathbf{J}_S^M\right|)$. With $\alpha = 0.6$, $v^* = 2.059$ and $\gamma = 0.85$, $(\alpha v - \gamma) > 0$ holds. The Jacobians of the subsystems and their determinants on the other hand become

$$\mathbf{J}_S^A = \begin{bmatrix} -0.223 & 0.743 \\ -0.193 & 0.223 \end{bmatrix}, \quad Det(\mathbf{J}_S^A) = 0.0938$$

$$\mathbf{J}_S^M = \begin{bmatrix} 0.0219 & -0.211 \\ 0.231 & -0.0219 \end{bmatrix}, \quad Det(\mathbf{J}_S^M) = 0.0482$$

The values indicate that all the stability conditions should remain satisfied regardless of the value of β, as $\alpha v > \gamma$ and $\left|\mathbf{J}_S^A\right| > \left|\mathbf{J}_S^M\right|$ so that accelerator effect on investment is larger than the profit rate effect and the multiplier-accelerator subsystem has a higher frequency. Note that this is only because of the chosen parameter values that satisfy $\alpha v > \gamma$ and if this condition is reversed while the Minsky cycle has a lower frequency (i.e $\left|\mathbf{J}_S^A\right| > \left|\mathbf{J}_S^M\right|$), the combination of two cycling mechanisms will be inherently destabilizing and the results we report below will also be reversed.

Figures 7-10 above present the values of $Tr(\mathbf{J})$, $Det(\mathbf{J})$, $|J_1| + |J_2| + |J_3|$ and $Det(\mathbf{J}) - Tr(\mathbf{J})(|J_1| + |J_2| + |J_3|)$ and confirm our conjecture in the theorem above that the combination of these subsystems will be inherently stabilizing for any $\beta \in [0, 1]$. Further, as differentiating (B28) in the Appendix with respect to β would suggest, the value of the stability condition $Tr(\mathbf{J})(|J_1| + |J_2| + |J_3|) + Det(\mathbf{J})$ peaks at $\beta = 0.5$. In other words, the stabilizing effect is at its peak when both systems have equal weight in the investment function, as intuition would also suggest. This is confirmed by Figure 11 where we depict the real parts of the eigenvalues for $\beta \in [0, 1]$ and by Figure 12B where we plot the solutions to the combined system for $\beta = 0.2$ and $\beta = 0.5$ and $\beta = 0.8$. The figures indicate that as the negative real part of the imaginary eigenvalues gets its largest absolute value at $\beta = 0.5$ with both systems having the same weight in the investment function, the combined system stabilises fastest towards the fixed point.

[11] This implies that g peaks before y in the multiplier-accelerator subsystem and before d in the Minsky subsystem.

Figure 12A: Growth Rate of Investment for different values of β

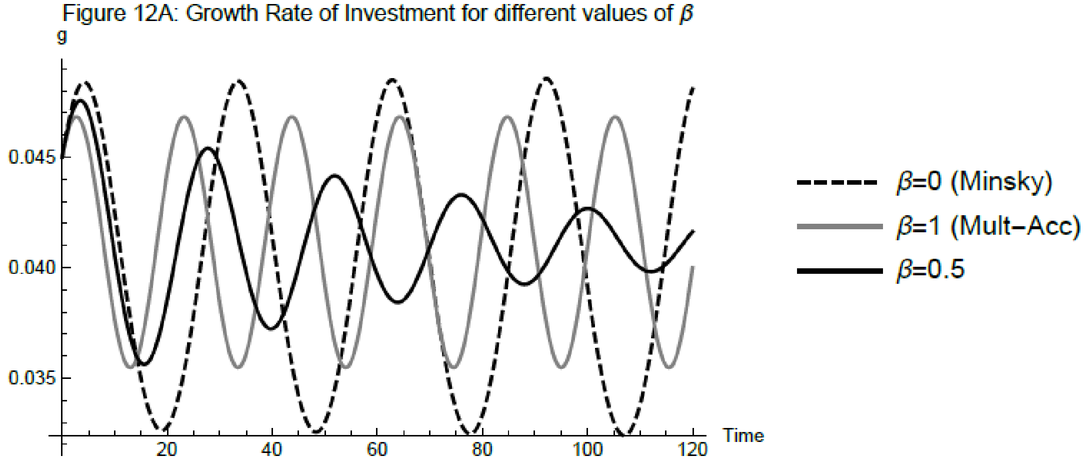
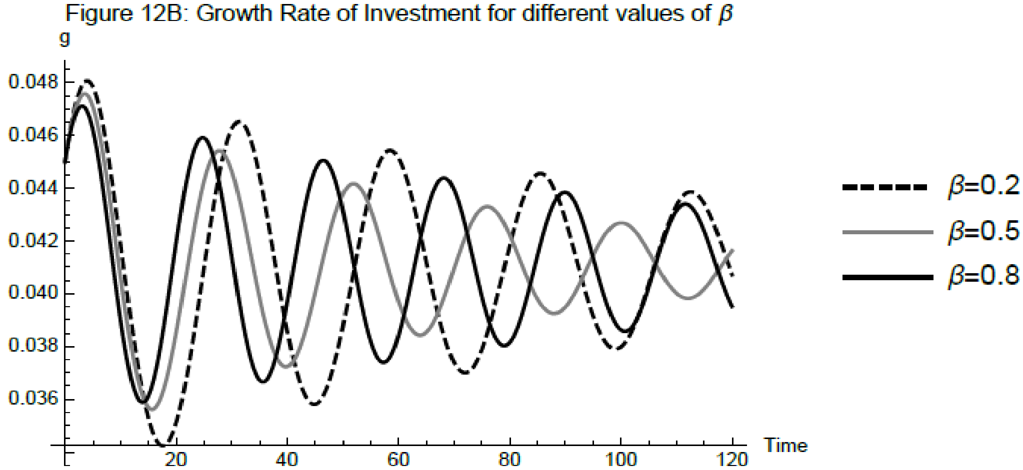


Figure 12B: Growth Rate of Investment for different values of β



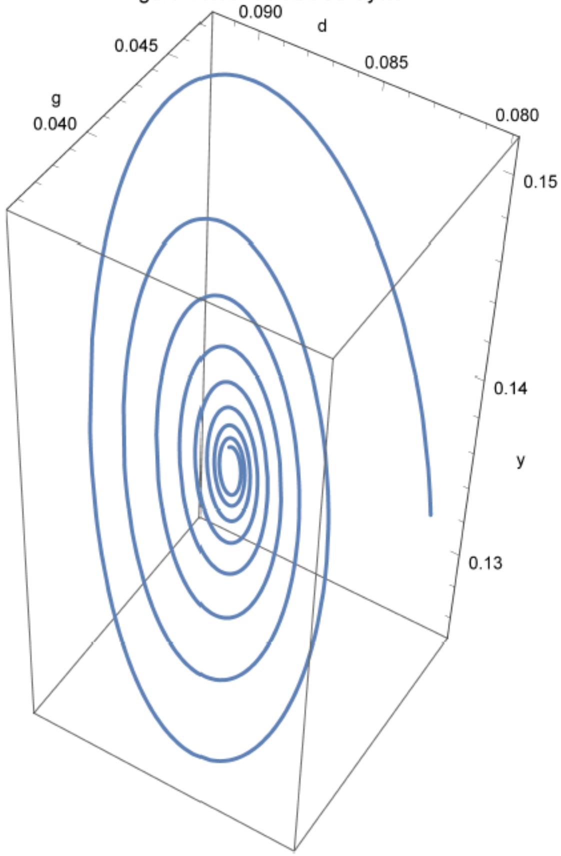


Figure 13:3D Combined Cycle

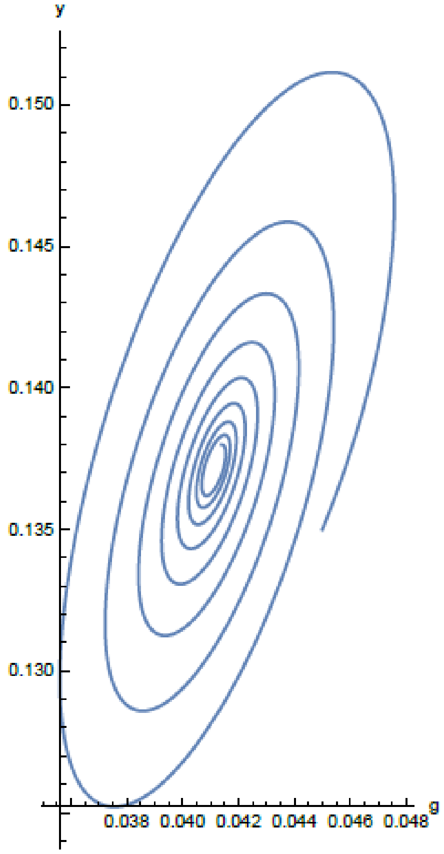


Figure 14:2D Combined Cycle in y–g

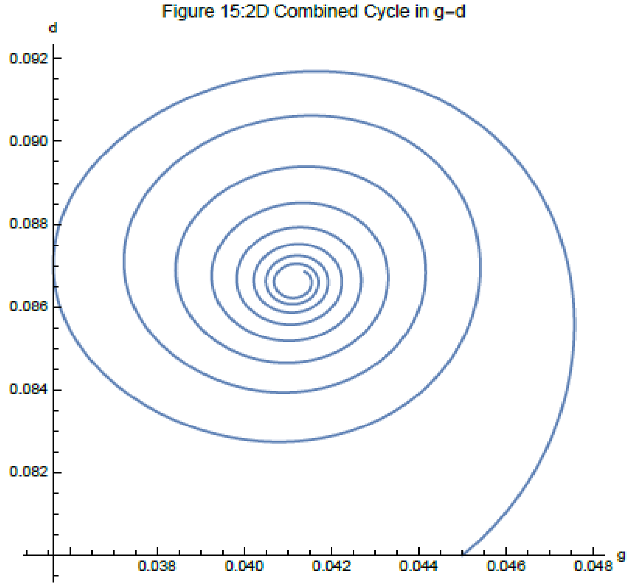


Figure 15:2D Combined Cycle in g–d

On the other hand, the dynamics of y, g and d in the combined system also show that the subsystem dynamics between $y-g$ and $d-g$ outlined above still operate when both cycle mechanisms are present. As Figure 14 and Figure 15 display, y and g converge to the fixed point with a dampening counter-clockwise cycle where g peaks first, as in the multiplier-accelerator subsystem.[12] Similarly, d and g also exhibit a counter-clockwise cyclical convergence to the fixed point with g peaking first, as in the Minsky subsystem. Therefore, the combined system preserves subsystem dynamics but the different effects of change in output and profit rate on investment and distinct periodicities of the subsystems always stabilise the combined model regardless of the weight of multiplier-accelerator and Minsky dynamics in investment.

5 Sensitivity Analysis

Since the combination of two models has an inherent stabilizing effect, we next increase v in order to make the multiplier-accelerator subsystem unstable. Figure 16 shows the value of the real parts of the complex eigenvalues as v increases from its critical value v^* given in Proposition 2. As expected, the more unstable the multiplier-accelerator system becomes, the higher the destabilizing effect of it on the combined system becomes compared to the stabilizing effect inherent in the model when both cycle mechanics are operating. The graph shows that for values of v such that $v^* < v < v^{\max}$, there is a value β^F at which a Hopf bifurcation emerges and the combination of an unstable multiplier-accelerator system with an asymptotically stable closed-orbit Minsky subsystem yields locally asymptotically stable closed orbits for the combined system. Emergence of closed orbits requires a lower relative weight of the unstable accelerator part in the investment function (lower β^F) as the multiplier-accelerator subsystem becomes more unstable and beyond a value of $v > v^{\max}$, the combined system becomes unstable regardless of the value of β and displays divergence via increasing oscillations.

We next move on to analyzing the stability of the combined system with respect to the speed of adjustment for investment (ϵ). Note that unlike v, increasing ϵ destabilizes both multiplier-accelerator and Minsky systems. In generating Figure 17 below, we keep $v = v^*$ in order to isolate the effect of ϵ alone on the stability of the combined system. As the figure shows, when $\epsilon = \epsilon^*$ and $v = v^*$, we reproduce the figure above, as both subsystems have closed orbits and their combination is always stable with negative real parts of the imaginary eigenvalues[13]. Increasing ϵ makes the real parts of the eigenvalues of both the Minskyian system positive at $\beta = 0$ and the eigenvalues of the multiplier-accelerator system positive at $\beta = 1$. For some values of ϵ,the combined system goes through a Hopf bifurcation twice. However, as ϵ increases further and both subsystems become more unstable, their combination also becomes always unstable despite the inherent stabilizing effect in the combined model.

[12] In Figures 13-15, $\beta = 0.5$ so that the stabilizing effect is at its maximum.

[13] Note that in all the simulations, the first two eigenvalues are complex and the third eigenvalue remains negative. Therefore, stability of the combined system depends on the real parts of the imaginary eigenvalues we report. Although we do not report the other stability conditions to save space, it is important to stress that $Det(J) \neq 0$ when the real part of the eigenvalues are zero in the graphs. In other words, the real eigenvalue does not become zero throughout our simulations.

Figure 16: Real Parts of Complex Eigenvalues for Different Values of v

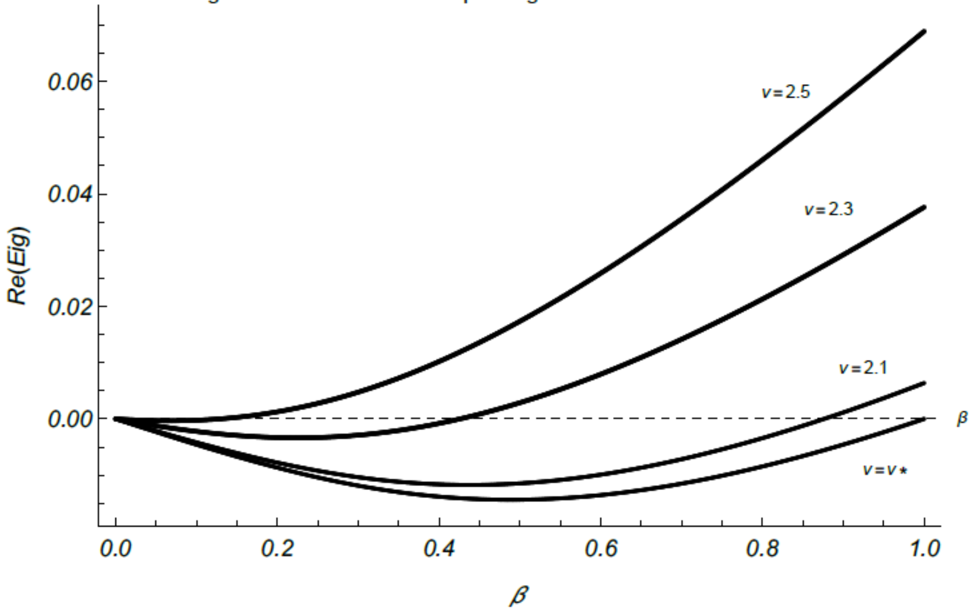


Figure 17: Real Parts of Complex Eigenvalues for Different Values of ϵ

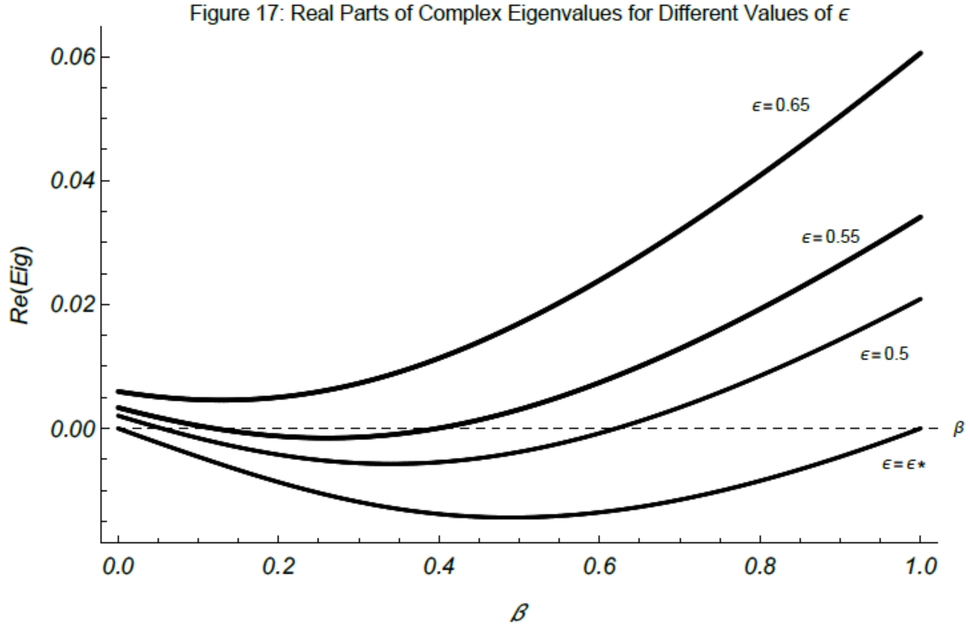


Figure 18: Growth Rate of Investment

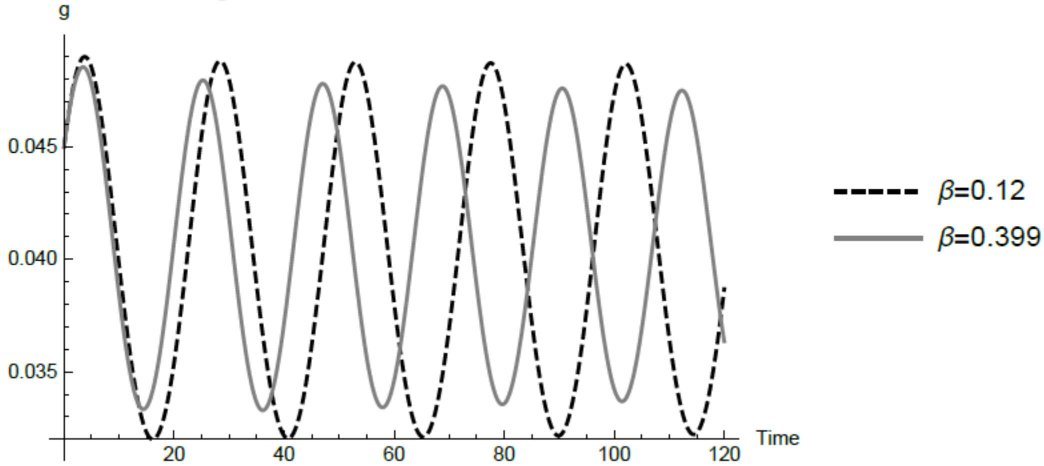


Figure 19: Real Parts of Complex Eigenvalues for Different Values of γ

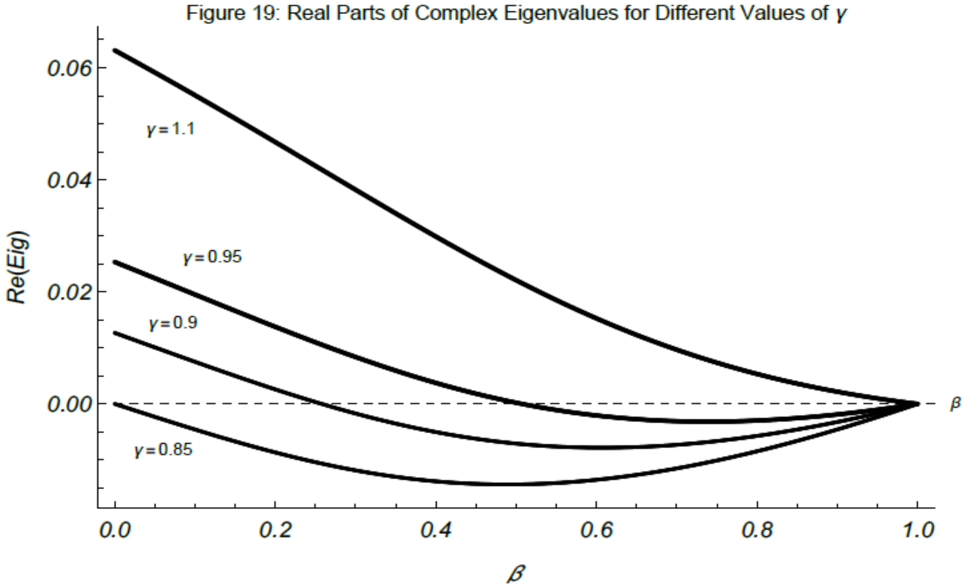
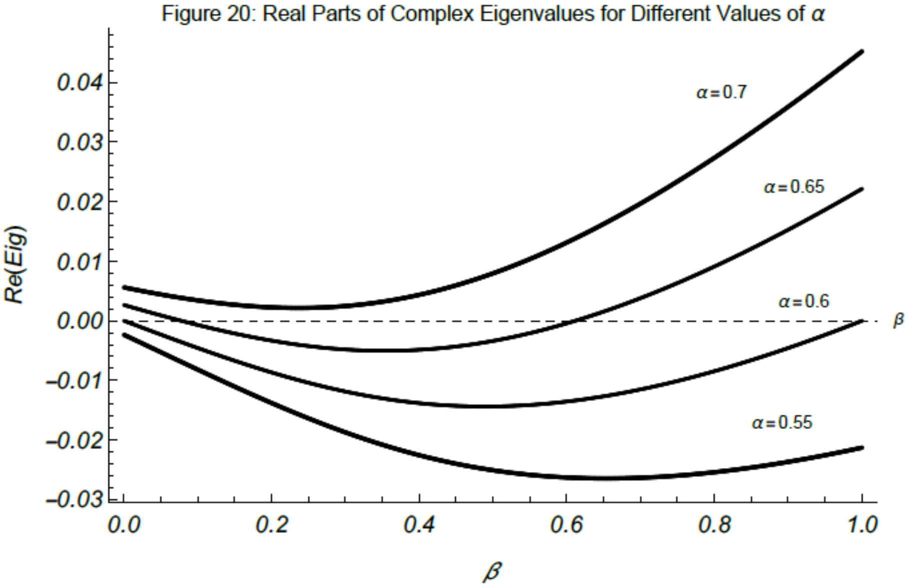


Figure 20: Real Parts of Complex Eigenvalues for Different Values of α



Let us now analyze the case $\epsilon = 0.55$ and $v = v^*$. Solving for the two bifurcation points on Figure 17 gives $\beta = 0.12$ and $\beta = 0.399$. In Figure 18 above, we plot the combined system for these two different values of β in order to analyze these closed orbits.The figure suggests that the magnitude and the frequency of the cycles differ at the two bifurcation points. With these values, the multiplier-accelerator subsystem's angular speed is larger and therefore its frequency is higher too. As a result, when the weight of the multiplier-accelerator is larger in the combined system (i.e $\beta = 0.399$), its frequency is also higher. Similarly, the multiplier-accelerator system has smaller cycles in magnitude and therefore for a larger share of the accelerator in the investment function, the cycles are also smaller.

Figure 19 on the other hand confirms that under Assumption 3, the negative relationship between the sensitivity of investment to profit rate and the stability of the Minsky model holds. Since γ does not change the multiplier-accelerator model, increasing this parameter from its initial value of 0.85 leaves the right hand side of the figure same, When $\beta = 1$, we have the multiplier-accelerator model, for which the real parts of the complex eigenvalues are zero since v is set at v^*. Once again, A Hopf bifurcation is possible with an unstable Minsky system and an multiplier-accelerator system with asymptotically stable closed orbits unless γ is too high and therefore the Minsky model is too unstable. Figure 19 thus displays the possibility of an unstable financial cycle destabilizing an asymptotically stable closed orbit real cycle despite the inherent stabilizing effect of simultenous operation of both cycle mechanisms; a case which might be of interest when considered in tandem with the increasing magnitude of the financial cycle documented in Borio (2012).

Figure 20 shows the same dynamics for the adjustment speed of the goods market (α). In order to generate Figure 20, we keep ϵ at ϵ^* as above so that we can isolate the affect of α alone on the stability of the subsystems. As with ϵ, an increase in the goods market adjustment speed destabilises both multiplier-accelerator and Minsky subsystems, therefore destabilizing the combined system too. Beyond a certain value of α, both subsystems are too unstable and there is no value of β for which their combination will yield asymptotically stable closed orbits any more. Our simulations also confirm the conjecture of Corollary 1 above that any parameter change that destabilizes one of the subsystems also destabilizes the combined system.

6 Conclusion

Using a three-dimensional model in continuous time, we have demonstrated that in the presence of two cycle mechanisms operating at the same time in an economy where these mechanisms share a self-destabilizing common variable, the stability of the overall economy depends on the frequencies of these sub-cycle mechanisms and the relative magnitude of the self-destabilizing effect of the common variable in each sub-system. In other words, coupling two stable sub-cycle mechanisms displaying closed orbits in isolation may create an overall unstable economy unless the sub-cycles are of the same frequency. Further, particularly marginally stable/unstable sub-systems may generate varying types of aggregate behaviour for the economic system, from explosive dynamics to closed orbits and stability depending on the frequencies of the sub-cycles. Our finding that the the frequencies of individual cycle mechanisms play a vital role in determining aggregate stability in the presence of multiple cycle mechanisms is particularly important for models such as Ryoo et al (2010) where there are short and long cycles operating at the same time in the economy. Our results suggest that the interaction of these cyces and the frequencies of short and long cycles should have an impact on the aggregate stability of the economy in such models.

From a policy perspective, our findings highlight the importance of correctly identifying sub-cycle mechanisms and their interactions, particularly if one interprets the accelerator part of our model as the real cycle and its Minskyian part as the financial cycle. The lower frequency of the financial cycle documented in Borio (2012), combined with a large profit rate effect on investment for instance may imply a destabilizing tendency in the economy, although individual real and financial cycle mechanisms may display stability in observed data. In essence, the financialization process may lead to an increase in the effect of profit rate on investment, thus destabilizing the aggregate economic system and necessitating prompt action from the government and the central bank during early phases of the cycle in order to mitigate and if possible, prevent the amplification of the cycle using appropriate policy tools. Such strong non-linear responses by policy makers, which are absent in our formulation, might be essential to ensure mild fluctuations around desirable levels for various policy variables and to prevent explosive dynamics.

Throughout the paper, we have assumed that the relative weights of two sub-cycle mechanisms on investment behaviour does not change. However, there is no reason why this should be the case. In fact, endogenous changes in the relative strength of these two effects on investment may create chaotic behaviour around a trend, with periods of relative calmness that resemble smooth convergence followed by explosive dynamics. An interesting topic for future research would therefore be to analyse the behaviour of the system when the weight parameter is endogenous and possibly subject to manipulation by policy.

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Appendix A

A1. Existence of fixed points of the multiplier-accelerator system

From (15), the single fixed point for the growth rate of investment g^* is given by g_0. Plugging this into (21), we find that $y^* = g_0/s_c\pi$.

The analysis of the third fixed point is more complicated due to the non-linerities in the dynamics of debt-capital ratio in (19). Since this equation is common in both the Multiplier and Minsky models, we will analyze its properties in detail. Setting (19) to zero and simplifying gives

$$d = \frac{\phi\Sigma^2 g^2 + (1-\Sigma)g}{(1+\phi\Sigma i)g - i} \tag{A1}$$

Rewriting this equation first as

$$d = \frac{\dfrac{\phi\Sigma^2}{(1+\phi\Sigma i)}g^2 + \dfrac{(1-\Sigma)}{(1+\phi\Sigma i)}g}{g - \dfrac{i}{(1+\phi\Sigma i)}} \tag{A2}$$

and then as

$$d = \frac{\dfrac{\phi\Sigma^2}{(1+\phi\Sigma i)}g\left[g - \dfrac{(\Sigma-1)}{\phi\Sigma^2}\right]}{g - \dfrac{i}{(1+\phi\Sigma i)}} \tag{A3}$$

we can see that there is an asymptote at $g = \dfrac{i}{(1+\phi\Sigma i)}$. Further, the value of $\Sigma - 1$ and $\dfrac{i}{(1+\phi\Sigma i)}$ determine the shape and behaviour of the function. In order to see this, rewrite the equation above as

$$d = \frac{\sigma_1 g\,[g-\sigma_2]}{g-\sigma_3} = f(g) \tag{A4}$$

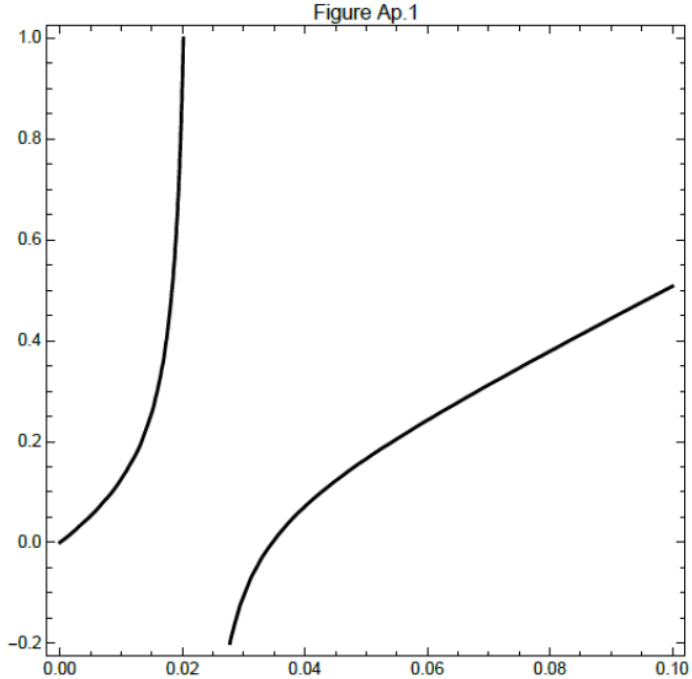
where $\sigma_1 = \dfrac{\phi\Sigma^2}{(1+\phi\Sigma i)} > 0, \sigma_2 = \dfrac{(\Sigma-1)}{\phi\Sigma^2} > 0$,and $\sigma_3 = \dfrac{i}{(1+\phi\Sigma i)} > 0$.

Since $\Sigma > 1$ and therefore $\sigma_2 > 0$, as well the origin, the function crosses the g axis also at $g = \sigma_2$.When g is above this value, the numerator is positive; when it is below this value the numerator is negative. Similarly, the denominator changes sign at $g = \sigma_3$ so the shape of the function will be determined by the relative sizes of σ_2 and σ_3.

Case 1: $\sigma_2 > \sigma_3$

In this case, the point at which the function crosses the g-axis is above the value at which there is an asmyptote. Therefore, for $g < \sigma_3$, both the numerator and the denominator is negative and d is positive while the function is increasing in g as the denominator gets close to σ_3. At $g = \sigma_3$, there's an asymptote to positive infinity. On the other hand, between $\sigma_3 < g < \sigma_2$, the numerator is negative but the denominator is positive, so the value of the function is negative and it is increasing in g. At $g = \sigma_2$, the function passes from the origin and then increases without bounds, as $lim_{g \to \infty} f(g) \to \infty$ and the slope of the curve converges to σ_1 as $g \to \infty$[14]. The shape of the curve in this case is depicted in Figure Ap.1 below and in Figures 1-2 in the text.

Figure Ap.1



Case 2: $\sigma_3 > \sigma_2$

In this case, the value at which the function becomes zero is lower than the value of g at which there is an asymptote. Therefore, for $g < \sigma_2$, the numerator and the denominator

[14] $f(g) = \frac{\beta_1 g[g-\beta_2]}{g-\beta_3}$ can be written as $f(g) = \frac{\beta_1 g - \beta_1 \beta_2}{1-\beta_3/g}$. As g goes to infinity, the slope of this function converges to β_1.

are both negative and the function is positive and first increasing then decreasing in g since at $g = \sigma_2$, the function crosses the g-axis as d becomes zero. In order to determine the behaviour of the function to the right of σ_2, let us take the derivative of $f(g)$.

$$f'(g) = \frac{(2\sigma_1 g - \sigma_1\sigma_2)(g - \sigma_3) - \sigma_1 g^2 + \sigma_2 g}{(g - \sigma_3)^2} \tag{A5}$$

Simplifying, we get

$$f'(g) = \frac{g^2 - 2\sigma_3 g + \sigma_2\sigma_3}{(g - \sigma_3)^2} \tag{A6}$$

Therefore, the sign of the derivative depends on the sign of the numerator, which can be written as

$$(g - \sigma_3)^2 + \sigma_2\sigma_3 - {\sigma_3}^2 \tag{A7}$$

If $\sigma_2\sigma_3 > {\sigma_3}^2$, which requires $\sigma_2 > \sigma_3$, the derivative is always positive and the function is always increasing. This is Case 1(a). If $\sigma_2 < \sigma_3$ on the other hand, (A7) can be written as

$$(g - \sigma_3)^2 - a \tag{A8}$$

where $a = {\sigma_3}^2 - \sigma_2\sigma_3 > 0$. Therefore, for $g > \sigma_3$, the sign of the derivative will change at the point

$$g^{\min} = \sigma_3 + \sqrt{a} > \sigma_3 \tag{A9}$$

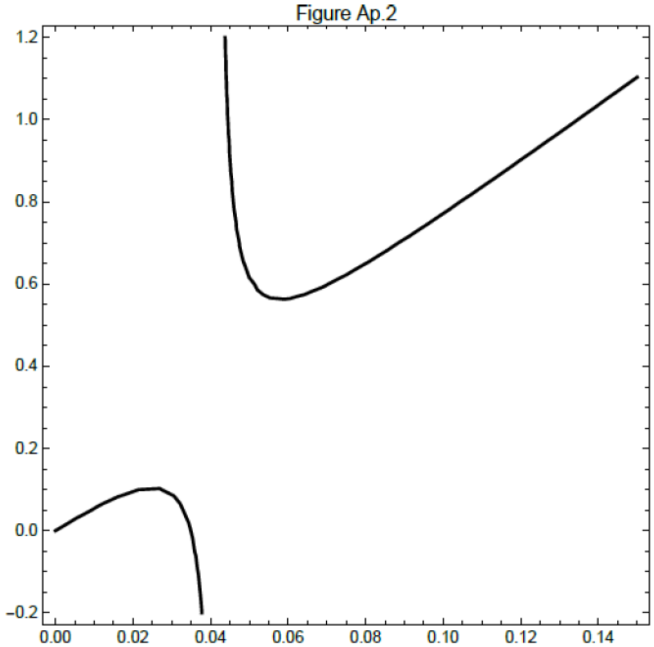
This means that to the right of the asymptote at $g = \sigma_3$, the function is first decreasing and after the minimum it reaches at $g^{\min}$, it increases without bounds, as $lim_{g\to\infty} f(g) \to .\infty$. This is depicted in the Figure Ap.2 below.

In both cases, given the value of g_0, there is a unique value for d^* which corresponds to it. Whether or not this value will be positive depends on the assumptions and parameter choices.

A2. Existence of fixed points of the Minsky System

The dynamic of motion for debt-capital ratio is common in the multiplier-accelerator and Minsky models, shape of the phase diagram depends on the values of σ_2 and σ_3. In both Case 1 and Case 2 outlined above, Minskyian system may have one more set of fixed points. In order to see this, note that as in the multiplier-accelerator model, the two-dimensional sub-system in g and d can be solved seperately with (23) and (24), and y can subsequently be found using (20). Setting (23) to zero gives us the nullcline for $\dot{g}$ as $d = g_0/\lambda + (\gamma\Sigma - 1)/\lambda g = h(g)$. This is a positively sloped line with the intercept $g_0/\lambda \neq 0$.

Figure Ap.2



If $\sigma_2 > \sigma_3$ as in Case 1A above and the slope of $f(g)_{g\to\infty}$ is larger than the slope of this line, (i.e $\sigma_1 = \dfrac{\phi\Sigma^2}{(1+\phi\Sigma i)} > (\gamma\Sigma - 1)/\lambda$), the $h(g)$ line and the $f(g)$ curve will intersect twice in the first quadrant, and there will be two sets of fixed points, as shown in Figure 1. If on the other hand, the slope of $h(g)$ line is higher than the slope of $f(g)_{g\to\infty}$ (i.e $\dfrac{\phi\Sigma^2}{(1+\phi\Sigma i)} < (\gamma\Sigma - 1)/\lambda$), the Minsky system will have one fixed point in the first quadrant, as the $h(g)$ line will intersect the $f(g)$ curve once before the asymptote at $g = \sigma_3$.

Appendix B

B.1 Stability of the Multiplier-accelerator system

The first two eigenvalues can be calculated by finding the eigenvalues of the $(3,3)$ minor of the Jacobian matrix. The third eigenvalue is equal to $-(1+\phi\Sigma i)g_0 + i$.

$$\mathbf{J}_S^A = \begin{bmatrix} -\Omega s_c \pi & \Omega \\ -\epsilon \Omega s_c \pi v & \epsilon(\Omega v - 1) \end{bmatrix} \tag{B1}$$

$$Tr(\mathbf{J}_S^A) = -\Omega s_c \pi + \epsilon(\Omega v - 1) \tag{B2}$$

In order to find the determinant of this matrix, let us first multiply coloumn 2 with $s_c\pi$ and add it to coloumn 1. The determinant of $\mathbf{J}_S^A$ is equal to the determinant of the new matrix

$$E = \begin{bmatrix} 0 & \Omega \\ -\epsilon s_c \pi & \epsilon(\Omega v - 1) \end{bmatrix}$$

which can easily be found as

$$\left|\mathbf{J}_S^A\right| = \Omega s_c \pi \epsilon > 0 \tag{B3}$$

Since the determinant is always positive, the stability of the system depends on the trace. Setting $Tr(\mathbf{J}_S^A) = 0$, we find the critical value of v as

$$v^* = \frac{\epsilon + \Omega s_c \pi}{\epsilon \Omega} > 0 \tag{B4}$$

For $v > v^*$ the system is unstable as $Tr(\mathbf{J}_S^A)$ becomes positive, and vice versa. The behavior of debt-capital ratio, however depends on the third eigenvalue, which is given by $-(1+\phi\Sigma i)g_0 + i$. Assumption 2 therefore ensures the third eigenvalue is negative so stability is ensured for $v < v^*$.

At $v = v^*$, the system will go through a Hopf bifurcation and closed orbits will emerge.

In order to prove this, we need to confirm that the assumptions of Hopf Bifurcation theorem is satisfied at this value. Since we have shown that $\left|\mathbf{J}_S^A\right| > 0$ and $Tr(\mathbf{J}_S^A) = 0$ at $v = v^*$, this only requires proving that the $Tr(\mathbf{J}_S^A)$ is not stationary with respect to v at v^*.

Differentiating $Tr(\mathbf{J}_S^A)$ with respect to v, we get

$$dTr(\mathbf{J}_S^A)/dv_{|v=v^*} = \epsilon\Omega \neq 0 \tag{B5}$$

unless $\epsilon = 0$ and/or $\Omega = 0$.

B.2 Stability of the Minsky System

To analyze the stability of the Minskyian subsystem, let us first restate the Jacobian matrix evaluated at the steady state:

$$\mathbf{J}_S^M = \begin{bmatrix} \epsilon(\gamma\Sigma - 1) & -\epsilon\lambda \\ 1 + 2\phi\Sigma^2 g_M - d_M^* - \Sigma - \phi\Sigma i d_M & -(1 + \phi\Sigma i)g_M^* + i \end{bmatrix}$$

While the Trace of this matrix is given by $\epsilon(\gamma\Sigma - 1) - (1 + \phi\Sigma i)g_M^* + i$ and its sign depends on the assumption on the value of $\epsilon(\gamma\Sigma - 1)$, the high level of non-linearity in the debt dynamics makes an algebraic calculation of the determinant of the subsystem cumbersome so we will instead present a graphical analysis of the stability of the fixed points.

Recall that the Minskyian subsystem is given by

$$\dot{g} = \epsilon\left[g_0 + (\gamma\Sigma - 1)g - \lambda d\right] = H(g, d)$$
$$\dot{d} = g(1 - d) - (1 - \phi\Sigma g)(\Sigma g - id) = F(g, d)$$

Denoting the partial derivative of any variable x with respect to F as F_x, the Jacobian above can be written as

$$\mathbf{J}_S^M = \begin{bmatrix} H_{g|H(g,d)=0} & H_{d|H(g,d)=0} \\ F_{g|F(g,d)=0} & F_{d|F(g,d)=0} \end{bmatrix} \tag{B6}$$

While we were analysing the existence of fixed points in the Minsky subsystem above, we had defined the nullcline for nullcline for $\dot{g}$ as $d = g_0/\lambda + (\gamma\Sigma - 1)/\lambda g = h(g)$ and the nullcline for $\dot{d}$ as $d = \frac{\sigma_1 g[g-\sigma_2]}{g-\sigma_3} = f(g)$. Therefore, for $H(g, d) = 0$, we have $d = h(g)$, and for $F(g, d) = 0$ we have $d = f(g)$. This implies that there is a relationship between $H(g, d)$ and $h(g)$ such that

$$h_g = -\frac{H_{g|H(g,d)=0}}{H_{d|H(g,d)=0}} \tag{B7}$$

Therefore, the slope of the $h(g)$ line is given by the negative of the ratio of a_{11} to a_{12} element of the Jacobian of the Minskyian subsystem. Similarly,

$$f_g = -\frac{F_{g|F(g,d)=0}}{F_{d|F(g,d)=0}} \tag{B8}$$

also holds and the ratio of a_{21} to a_{22} element of the subsystem's Jacobian gives the slope of the $f(g)$ curve.

Assume now (without loss of generality) that $\sigma_2 > \sigma_3$ and $\sigma_1 = \dfrac{\phi\Sigma^2}{(1 + \phi\Sigma i)} > (\gamma\Sigma - 1)/\lambda$ so that the phase diagram of the Minsky system looks like in Figure 1. The system has two sets of fixed points. There is a fixed point with a low growth $g_M^* < \sigma_3$ and another one with high growth $g_M^* > \sigma_2$.

Let us first consider the low growth fixed point with $g_M^* < \sigma_3$. As Figure 1 shows, both $h(g)$ and $f(g)$ are positively sloped at the intersection point. Further, note that the a_{22} element of the subsystem's Jacobian, $-(1+\phi\Sigma i)g_M^* + i$, becomes negative only if $g_M^* > \frac{i}{(1+\phi\Sigma i)} = \sigma_3$. Therefore, at this low growth fixed point, the subsystem's Jacobian's elements take the following sign structure:

$$\mathbf{J}_S^M = \begin{bmatrix} H_g(+) & H_d(-) \\ F_g(-) & F_d(+) \end{bmatrix} \tag{B9}$$

The sign of the determinant cannot be determined from this sign structure. However, since from Figure 1 we can see that $h(g)$ line cuts the $f(g)$ curve from above, its slope must be lower than the slope of $f(g)$. This implies

$$-\frac{F_g}{F_d} > -\frac{H_g}{H_d} \tag{B10}$$

With $F_d > 0$ and $H_d < 0$, this implies

$$F_g H_d > F_d H_g > 0 \Rightarrow \left|\mathbf{J}_S^M\right| < 0 \tag{B11}$$

which ensures that the fixed point is a saddle. If on the other hand $h(g)$ line is negatively sloped, then the sign structure of $\mathbf{J}_S^M$ becomes

$$\mathbf{J}_S^M = \begin{bmatrix} H_g(-) & H_d(-) \\ F_g(-) & F_d(+) \end{bmatrix} \tag{B12}$$

which ensures that $\left|\mathbf{J}_S^M\right| < 0$ always holds.

For the high growth fixed point with $g_M^* > \sigma_2 > \sigma_3$, a_{22} of the Jacobian of the Minskyian subsystem is negative, and therefore with a positively sloped $h(g)$ and $f(g)$, Jacobian elements' signs become

$$\mathbf{J}_S^M = \begin{bmatrix} H_g(+) & H_d(-) \\ F_g(+) & F_d(-) \end{bmatrix} \tag{B13}$$

Once again, the sign structure of this matrix does not determine the sign of $\left|\mathbf{J}_S^M\right|$ on its own. However, as the $h(g)$ line cuts the $f(g)$ curve from above to the right of the asymptote at σ_3 in Figure 1, it is flatter than $f(g)$.We again have

$$-\frac{F_g}{F_d} > -\frac{H_g}{H_d} \tag{B14}$$

With $F_d < 0$ and $H_d < 0$, this implies

$$\frac{F_g}{F_d} < \frac{H_g}{H_d} \Rightarrow F_g H_d < F_d H_g < 0, \tag{B15}$$

which ensures $\left|\mathbf{J}_S^M\right| > 0$ holds. Therefore, as conjectured in Proposition 4, if the $h(g)$ line cuts the $f(g)$ curve from above at a point below the asymptote at $g = \sigma_3$, $\left|\mathbf{J}_S^M\right| < 0$ and the fixed point is a saddle. If on the other hand the $h(g)$ line cuts the $f(g)$ curve from above at a point above the asymptote at $g = \sigma_3$, $\left|\mathbf{J}_S^M\right| > 0$ and the stability properties of the Minskyian subsystem depends on $Tr(\mathbf{J}_S^M)$.[15] Setting $Tr(\mathbf{J}_S^M) = 0$,we get the condition on ϵ for a Hopf bifurcation as stated in proposition 5. As above, it is straigtforward to show that $Tr(\mathbf{J}_S^M)/d\epsilon_{\epsilon=\epsilon^*} \neq 0$ so the bifurcation is not degenerate.

B.3 Proof of Theorem 1 and Corollary (Stability of the Fully-coupled System)

In order to prove the theorem, let us first assume that

$$S_1 = \begin{array}{c} \dot{x} = f(x,y) \\ \dot{y} = h(x,y) \end{array}$$

and

$$S_2 = \begin{array}{c} \dot{y} = p(y,z) \\ \dot{z} = q(y,z) \end{array}$$

have a common fixed point (x^*, y^*, z^*) such that $f(x^*, y^*) = h(x^*, y^*) = p(y^*, z^*) = q(y^*, z^*) = 0$. Then, (x^*, y^*, z^*) is also a fixed point for the system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \beta h(x,y) + (1-\beta)p(y,z) \\ \dot{z} &= q(y,z) \end{aligned}$$

since we can write

$$\dot{y} = \beta h(x,y) + (1-\beta)p(y,z)$$

as

$$\dot{y} = \beta[h(x,y) - p(y,z)] + p(y,z)$$

for which

$$\dot{y} = \beta[h(x^*,y^*) - p(y^*,z^*)] + p(y^*,z^*) = 0$$

always holds.

[15]A similar stability analysis can be carried out for the case $\sigma_2 < \sigma_3$ shown in Figure A.2 above for downward/upward sloping $h(g)$ lines. However, since our parameter calcibration satisfies $\sigma_2 > \sigma_3$ with upward-sloping $h(g)$ line for reasons mentioned in the text, we will not explicitly present stability conditions in these cases.

Define the Jacobians of the independent 2D subsystems S_1 and S_2 as

$$\mathbf{J}_{S_1} = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \tag{B16}$$

$$\mathbf{J}_{S_2} = \begin{bmatrix} e_5 & e_6 \\ e_7 & e_8 \end{bmatrix} \tag{B17}$$

while the Jacobian of the fully-coupled 3D system is given by

$$\mathbf{J} = \begin{bmatrix} e_1 & e_2 & 0 \\ \beta e_3 & \beta e_4 + (1-\beta)e_5 & (1-\beta)e_6 \\ 0 & e_7 & e_8 \end{bmatrix} = \begin{bmatrix} - & e_2 & 0 \\ \beta e_3 & + & (1-\beta)e_6 \\ 0 & e_7 & - \end{bmatrix} \tag{B18}$$

We do not need to assume a specific sign structure for e_2, e_3, e_6 or e_7.Stability of this three-dimensional system requires

$$Tr(\mathbf{J}) < 0, \;\; |\mathbf{J}| < 0, \quad |\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| > 0 \tag{B19}$$

$$\text{and} \quad -Tr(\mathbf{J})(|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3|) + |\mathbf{J}| > 0, \tag{B20}$$

where

$$\mathbf{J}_1 = \begin{bmatrix} \beta e_4 + (1-\beta)e_5 & (1-\beta)e_6 \\ e_7 & e_8 \end{bmatrix}, \tag{B21}$$

$$\mathbf{J}_2 = \begin{bmatrix} e_1 & 0 \\ 0 & e_8 \end{bmatrix}, \tag{B22}$$

$$\mathbf{J}_3 = \begin{bmatrix} e_1 & e_2 \\ \beta e_3 & \beta e_4 + (1-\beta)e_5 \end{bmatrix} \tag{B23}$$

are the principal minor matrices.

The trace of the fully-coupled system can be written as

$$Tr(\mathbf{J}) = e_1 + \beta e_4 + (1-\beta)e_5 + e_8 \tag{B24}$$

$$Tr(\mathbf{J}) = Tr(\mathbf{J}_{S_1}) + Tr(\mathbf{J}_{S_2}) - [\beta e_5 + (1-\beta)e_4] \tag{B25}$$

If both subsystems are stable, $Tr(\mathbf{J}_{S_1}) < 0$ and $Tr(\mathbf{J}_{S_2}) < 0$ hold. Since $\beta < 1$ and $e_1 < 0$, $e_1 + \beta e_4 < 0$ regardless of the sign of e_4, and $(1-\beta)e_5 + e_8 < 0$. Therefore, $Tr(\mathbf{J}) < 0$ holds if the subsystems are stable.

On the other hand, if both systems are unstable, $e_4 > e_1$ and $e_8 > e_5$, and $\beta < 1$ implies that $e_1 + \beta e_4 \lesseqgtr 0$ and $(1-\beta)e_5 + e_8 \lesseqgtr 0$. Therefore, the sign of the $Tr(\mathbf{J})$ is ambigous.

In order to find an expression for $|\mathbf{J}|$, let us write

$$|\mathbf{J}| = e_1 \begin{vmatrix} \beta e_4 + (1-\beta)e_5 & (1-\beta)e_6 \\ e_7 & e_8 \end{vmatrix} - e_2 \begin{vmatrix} \beta e_3 & (1-\beta)e_6 \\ 0 & e_8 \end{vmatrix}$$

$$|\mathbf{J}| = e_1 \left[\beta e_8 e_4 + (1-\beta)e_5 e_8 - (1-\beta)e_6 e_7\right] - e_2 \beta e_3 e_8$$

$$|\mathbf{J}| = e_1 \left[\beta e_8 e_4 + (1-\beta)\left|\mathbf{J}_{S_2}\right|\right] - e_2 \beta e_3 e_8$$

$$|\mathbf{J}| = e_1 (1-\beta)\left|\mathbf{J}_{S_2}\right| + e_8 \beta (e_1 e_4 - e_2 e_3)$$

$$|\mathbf{J}| = e_1 (1-\beta)\left|\mathbf{J}_{S_2}\right| + e_8 \beta \left|\mathbf{J}_{S_1}\right| \tag{B26}$$

So the determinant of the combined system is a weigthed average of the determinants of the subsystems scaled by two diagonal elements, e_1 and e_8. If both subsystems are stable, $|\mathbf{J}_{S_2}| > 0$ and $|\mathbf{J}_{S_1}| > 0$. Since $e_1 < 0$ and $e_8 < 0$, $|\mathbf{J}| < 0$ holds and the stability condition is satisfied. Similarly, the combination of two systems with saddle points will lead to $|\mathbf{J}| > 0$ and result in an combined system with a saddle point, as stated in the theorem.

The determinants of the principal minors of the Jacobian of the combined system can be found as follows.

$$\mathbf{J}_1 = \begin{bmatrix} \beta e_4 + (1-\beta)e_5 & (1-\beta)e_6 \\ e_7 & e_8 \end{bmatrix}$$

$$|\mathbf{J}_1| = e_8 \beta e_4 + (1-\beta)e_5 e_8 - (1-\beta)e_6 e_7$$

$$|\mathbf{J}_1| = \beta e_8 e_4 + (1-\beta)\left|\mathbf{J}_{S_2}\right|$$

$$\mathbf{J}_2 = \begin{bmatrix} e_1 & 0 \\ 0 & e_8 \end{bmatrix}$$

$$|\mathbf{J}_2| = e_1 e_8$$

$$\mathbf{J}_3 = \begin{bmatrix} e_1 & e_2 \\ \beta e_3 & \beta e_4 + (1-\beta)e_5 \end{bmatrix}$$

$$|\mathbf{J}_3| = \beta e_1 e_4 + (1-\beta)e_1 e_5 - \beta e_2 e_3$$

$$|\mathbf{J}_3| = \beta\,|\mathbf{J}_{S_1}| + (1-\beta)e_1 e_5$$

$$|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| = \beta\,|\mathbf{J}_{S_1}| + (1-\beta)\,|\mathbf{J}_{S_2}| + e_1 e_8 + \beta e_8 e_4 + (1-\beta)e_1 e_5$$

If both systems are stable, $|\mathbf{J}_{S_1}| > 0$ and $|\mathbf{J}_{S_2}| > 0$. so the first two terms are positive while $e_1 e_8 > 0$, $\beta e_8 e_4 < 0$ when $e_4 > 0$ and $(1-\beta)e_1 e_5 < 0$. Further in this case, $|e_1 e_8| > |e_8 e_4|$ because stability of the S_1 requires $|e_1| > |e_4|$. Similarly, if S_2 is stable, $|e_8| > |e_5|$ and therefore $|e_1 e_8| > |e_1 e_5|$. Since $|e_1 e_8|$ is greater than both $|e_8 e_4|$ and $|e_1 e_5|$, $\beta < 1$ implies $|e_1 e_8| > |\beta e_8 e_4 + (1-\beta)e_1 e_5|$. So $|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| > 0$ always holds if both subsystems are stable. The proof is more straightforward for $e_4 < 0$ since $\beta e_8 e_4 < 0$ in this case and $|e_1 e_8| > |e_1 e_5|$ once again implies that $|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| > 0$.

The last stability condition requires that

$$-Tr(\mathbf{J})(|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3|) + |\mathbf{J}| > 0$$

$$-Tr(\mathbf{J})\left[\beta|\mathbf{J}_{S_1}| + (1-\beta)|\mathbf{J}_{S_2}| + e_1 e_8 + \beta e_8 e_4 + (1-\beta)e_1 e_5\right]$$

$$+e_1(1-\beta)|\mathbf{J}_{S_2}| + e_8\beta|\mathbf{J}_{S_1}| > 0$$

$$-Tr(\mathbf{J})\beta|\mathbf{J}_{S_1}| - (1-\beta)Tr(\mathbf{J})|\mathbf{J}_{S_2}| - Tr(\mathbf{J})\left[e_1 e_8 + \beta e_8 e_4 + (1-\beta)e_1 e_5\right]$$
$$+e_1(1-\beta)|\mathbf{J}_{S_2}| + e_8\beta|\mathbf{J}_{S_1}| > 0$$

$$\beta|\mathbf{J}_{S_1}|(e_8 - Tr(\mathbf{J})) + (1-\beta)|\mathbf{J}_{S_2}|(e_1 - Tr(\mathbf{J})) - Tr(\mathbf{J})\left[e_1 e_8 + \beta e_8 e_4 + (1-\beta)e_1 e_5\right] > 0$$

This equation can further be simplified as

$$-\beta|\mathbf{J}_{S_1}|\left[e_1 + \beta e_4 + (1-\beta)e_5\right] - (1-\beta)|\mathbf{J}_{S_2}|\left[e_8 + \beta e_4 + (1-\beta)e_5\right]$$
$$-Tr(\mathbf{J})\left[e_1 e_8 + \beta e_8 e_4 + (1-\beta)e_1 e_5\right] > 0$$

Denoting the last term as $Q = -Tr(\mathbf{J})\left[e_1 e_8 + \beta e_8 e_4 + (1-\beta)e_1 e_5\right]$,

$$-\beta|\mathbf{J}_{S_1}|\left[e_1 + e_4 - (1-\beta)e_4 + (1-\beta)e_5\right] - (1-\beta)|\mathbf{J}_{S_2}|\left[e_8 + \beta e_4 + e_5 - \beta e_5\right] + Q > 0$$

$$\beta|\mathbf{J}_{S_1}|\left[(1-\beta)(e_4 - e_5) - (e_1 + e_4)\right] - (1-\beta)|\mathbf{J}_{S_2}|\left[\beta(e_4 - e_5) + (e_5 + e_8)\right] + Q > 0$$

$$\beta(1-\beta)(e_4 - e_5)\left(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|\right) - \beta|\mathbf{J}_{S_1}|Tr(\mathbf{J}_{S_1}) - (1-\beta)|\mathbf{J}_{S_2}|Tr(\mathbf{J}_{S_2}) + Q > 0 \quad \text{(B27)}$$

must hold. If both subsystems are stable,

$$Q > 0, \quad \beta|\mathbf{J}_{S_1}|Tr(\mathbf{J}_{S_1}) < 0, \quad |\mathbf{J}_{S_2}|Tr(\mathbf{J}_{S_2}) < 0$$

hold. Therefore, the last three terms are positive. However, the sign of the first term is ambigous. So the combination of two stable subsystems may not yield a stable system, particularly if the stability of the subsystems are marginal. However, a sufficient condition as stated in the theorem for the stability of the combined system can be derived as $e_4 > e_5 \cup |\mathbf{J}_{S_1}| > |\mathbf{J}_{S_2}|$ or $e_4 < e_5 \cup |\mathbf{J}_{S_1}| < |\mathbf{J}_{S_2}|$.

Equation (B27) also shows that as both systems become more stable (i.e $Tr(\mathbf{J}_{S_1}), Tr(\mathbf{J}_{S_2})$ become more negative and Q becomes more positive), the combined system also becomes more stable.

When two systems with closed orbits are combined with each other, $Tr(J) < 0$ holds, $|J| < 0$ holds as above.

$$|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| = \beta|\mathbf{J}_{S_1}| + (1-\beta)|\mathbf{J}_{S_2}| + e_1 e_8 + \beta e_8 e_4 + (1-\beta)e_1 e_5$$

With both subsystems giving closed orbits, $|e_1| = |e_4|$ and $|e_5| = |e_8|$. Therefore,

$$|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| = \beta|\mathbf{J}_{S_1}| + (1-\beta)|\mathbf{J}_{S_2}| + e_1 e_8 - \beta e_8 e_1 - (1-\beta)e_1 e_8$$

$$|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3| = \beta|\mathbf{J}_{S_1}| + (1-\beta)|\mathbf{J}_{S_2}| > 0$$

As above, we can derive the last stability condition as

$$\beta(1-\beta)(e_4 - e_5)\left(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|\right) - \beta|\mathbf{J}_{S_1}|Tr(\mathbf{J}_{S_1}) - (1-\beta)|\mathbf{J}_{S_2}|Tr(\mathbf{J}_{S_2}) + Q > 0$$

With two subsystems with closed orbits, we have $Tr(\mathbf{J}_{S_1}) = 0, Tr(\mathbf{J}_{S_2}) = 0$ and $Q = 0$. So the equation boils down to:

$$\beta(1-\beta)(e_4 - e_5)\left(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|\right) \tag{B28}$$

Therefore, when equation (B28) crosses zero, the Routh-Hurwitz conditions for a Hopf bifurcation are satisfied and asymptotically locally stable closed orbits emerge.[16] From (B27), we can see that the combination of two subsystems with asymtotically stable closed orbits will give a stable system if $e_4 > e_5$ & $|\mathbf{J}_{S_1}| > |\mathbf{J}_{S_2}|$ or $e_4 < e_5$ & $|\mathbf{J}_{S_1}| < |\mathbf{J}_{S_2}|$. Further, this equation will be equal to zero only if $e_4 = e_5$ and/or $|\mathbf{J}_{S_1}| = |\mathbf{J}_{S_2}|$. In this case, the last stability condition is equal to zero regardless of the value of β. In other words, the stability condition is stationary with respect to β and the necessary condition for Hopf bifurcation is violated. The combination of two subsystems with the same closed-orbit fixed point will always yield the same fixed point with closed orbits around it for any $\beta \in [0,1]$ if $e_4 = e_5$ and/or $|\mathbf{J}_{S_1}| = |\mathbf{J}_{S_2}|$.

The derivative of the stability condition evaluated at any bifurcation value β^F becomes:

$$d\left[-Tr(\mathbf{J})(|\mathbf{J}_1| + |\mathbf{J}_2| + |\mathbf{J}_3|) + |\mathbf{J}|)\right]/d\beta_{|\beta=\beta^F} = (1-2\beta^F)(e_4 - e_5)\left(|\mathbf{J}_{S_1}| - |\mathbf{J}_{S_2}|\right)$$
$$-|\mathbf{J}_{S_2}|Tr(\mathbf{J}_{S_2}) + dQ/d\beta_{|\beta=\beta^F} \neq 0$$

unless by chance.

[16] A mathematical proof of the Routh-Hurwitz conditions using Orlando's Formula can be found in Gantmacher (1954:197). For a quick and intuitive way to establish the condition for a Hopf bifurcation, recall that the third order polynomial for the characteristic equation of a three-dimensional system is given by

$$\lambda^3 - Tr(J)\lambda^2 + (|J_1| + |J_2| + |J_3|)\lambda - |J| = 0$$

In order to have two imaginary roots and a negative root, which is the necessary condition for closed orbits, a third order polynomial must satisfy

$$ax^3 + bx^2 + cx + d = (x^2 + m)(x + n)$$

where $n > 0$ and $m > 0$.
Multiplying the right-hand side, we get

$$x^3 + nx^2 + mx + mn.$$

Comparing this equation with the characteristic equation shows that $Tr(J) < 0$ and $|J_1| + |J_2| + |J_3| > 0$ must hold. Since the last term $nm > 0, |J| < 0$ must also hold. And finally, the coefficents in front of x^2 and x must multiply to give the last term, implying that

$$-Tr(J)|J_1| + |J_2| + |J_3| = -|J|,$$

or rather in its more widely used form

$$-Tr(J)|J_1| + |J_2| + |J_3| + |J| = 0.$$

Therefore, when the value of $Tr(J)|J_1| + |J_2| + |J_3| + |J|$ crosses zero while $Tr(J) < 0, |J| < 0$,$|J_1| + |J_2| + |J_3| > 0$ hold, the real part of imaginary roots disappear while the third root is negative and a Hopf bifurcation emerges. For the bifurcation to be non-degenerate, the derivative of the condition $-Tr(J)|J_1| + |J_2| + |J_3| + |J|$ with respect to the bifurcation parameter evaluated at the bifurcation point should be non-zero. See Asada and Yoshida (2003) for a similar approach to the 4D case.

Finally, note that in a two dimensional system with pure imaginary eigenvalues, $\lambda = \pm\frac{\sqrt{|\Delta|}}{2}i$ where $|\Delta| = |\mathbf{Jc}|$ where $\mathbf{Jc}$ is the Jacobian matrix of the system evaluated at the fixed point. In such a case, $\Psi = \sqrt{|\mathbf{Jc}|}$gives the angular speed of the closed orbits which will repeat every $T = \frac{2\pi}{\Psi}$ units of time. Therefore, a higher determinant implies a higher angular speed and therefore a lower periodicity (higher frequency) with pure imaginary eigenvalues, as stated in the theorem.

In order to prove the last part of the theorem, let us assume $h_y < 0$ and $p_y < 0$ instead. The Jacobian of the 3D system now takes the form

$$J = \begin{bmatrix} e_1 & e_2 & 0 \\ \beta e_3 & \beta h_y + (1-\beta)p_y & (1-\beta)e_6 \\ 0 & e_7 & e_8 \end{bmatrix} = \begin{bmatrix} + & e_2 & 0 \\ \beta e_3 & - & (1-\beta)e_6 \\ 0 & e_7 & + \end{bmatrix} \tag{B29}$$

Recall that the determinant of the 3D system is given by

$$|J| = e_1(1-\beta)Det(S_1) + e_8\beta Det(S_2) > 0 \tag{B30}$$

will always hold, implying that the fixed point y^* will be a saddle point.

Note that the condition $e_4 > e_5$, stated in the corollary to Theorem 1 as $\alpha v > \gamma$, requires

$$\epsilon(\Omega v - 1) > \epsilon(\gamma\Sigma - 1)$$

Using $\Omega = \dfrac{\alpha}{(1-\alpha\tau m_c\pi.)}$ and $\Sigma = 1 + m_c\pi\Omega\tau$, we get

$$\frac{\alpha v}{1-\alpha\tau m_c\pi} > \gamma(1 + \frac{\tau m_c\pi\alpha}{1-\alpha\tau m_c\pi.})$$

Simplifying this inequality gives the condition in the Corollary:

$$\alpha v > \gamma$$

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