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A Minimal Probabilistic Minsky Model: 3D Continuous-Jump Dynamics

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Abstract: This paper proposes a formalization of Hyman P. Minsky's theory of financial instability. The model includes private-sector borrowing, capacity utilization, and the stock of private-sector debt. The model is based on self-reinforcing borrowing and output dynamics that repeatedly come to a sudden stop, with discontinuous downward jumps in the three variables. The paper treats as endogenous the instantaneous probability of a jump and the size distribution of jump vectors. Formally, the model comprises three ordinary differential equations and a compound Poisson process, with jumps drawn from a heavy-tailed stable distribution. The paper shows it can be stated in three equations in the jump differentials and the usual differentials. A section sketches a nonlinear mechanism that can bound the system. The paper analyzes the dynamics of a simplified version of the main model and a more-SFC model with feedbacks from debt to borrowing and capacity utilization via debt-service effects. The paper reports (1) eigenvalues for the linear parts of both the simplified analytical model and a numerical example of the more-SFC model, (2) a phase diagram for the analytical model, and, (3) analytical stability conditions for the more-SFC model. The model replicates the upward instability and abrupt crises of Minsky's theory.

Keywords: Minsky model, paradox of debt, Poisson process, financial crisis, dynamical macroeconomic model, Hyman P. Minsky, stable distribution, stock-flow consistency, theory of financial instability, dynamical systems, cádlág process, John Maynard Keynes, Michal Kalecki, Joan Robinson

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I. Modeling Minsky Probabilistically in Three State Variables

The exact timing of the onset of a financial crisis is random as far as we know. In a series of papers written alone and with a coauthor, the author has proposed a formalization of Minsky's theory of financial crises (Hannsgen 2012, 2013; Hannsgen and Young-Taft 2015, 2017).

Three key elements of this model were:

1) A tendency toward growth and financial expansion that develops spontaneously in the aftermath of a financial crisis.

2) A measure of financial risk or fragility that rises over time following the end of a financial crisis, in keeping with Minsky's key point that the financing structures of capitalism are conducive to the emergence of financial fragility (e.g., Minsky 1977, 66; 1986, 213).

3) a nonhomogeneous Poisson probability model (e.g., Ross 1997, 235–302, esp. 277–81; Walde 2011 225–238), in which a crisis is a jump in state variables that occurs in any interval of time with a probability that depends on economic and financial variables. Jumps occur within a model that generates otherwise smooth trajectories.[1]

As Minsky put it, "Whereas experimentation with extending debt structures can go on for years…the revaluation of acceptable debt structures, when something goes wrong, can be quite sudden" (1977, 67).

Discussion of the Literature: The model is closely related to many in the literature. In early work, Minsky (e.g., 1959) himself suggested a jump model of sorts in the form of a discrete-time multiplier-accelerator model that restarted at new initial conditions when it hit a ceiling or floor. Among the factors that could impel the economy to its ceiling would be a financial boom. For some parameter values and assumptions, the model could generate cycles.

The literature developing models based on Minsky's theory of financial instability is large. Surveys by Dos Santos (2005) and Nikolaidi and Stockhammer (2017) and an essay by Ferri (1992) provide overviews of what Dos Santos dubbed the "formal Minskyan literature" (2005, 711)

At least three early analytical looks at Minsky posited formal models (Taylor and O'Connell 1985; Lavoie 1986–87; Keen 1995). Taylor and O'Connell (1985) emphasized the role of interest and profit rates in an early effort to integrate Minsky into a structuralist growth model. Lavoie (1986–87) sharply criticized the standard Minskyan story in which inevitably rising interest rates play a part in a fall in the rate of profit, leading to crisis. As he showed, this profit-squeeze version of Minsky assumes a great deal about the dynamics of other variables. Keen (1995, 2011) incorporated key elements of Minsky (1977, 1986) and Goodwin (1967) into a multidimensional, continuous-time model of growth, distribution, and employment.

Subsequent efforts incorporated endogenous money with interest-rate-setting by the central bank or the financial sector. Of course, Minsky himself (e.g., 1986, 131) opposed the standard IS-LM model and

[1] Poisson jump processes were deployed in the neoclassical version of Schumpeterian endogenous growth theory (Aghion and Howitt 1997, 55).

its assumption of an exogenous stock of money.[2] Building from Taylor's paper, Foley (2003) constructed an open-economy, Kaleckian Minsky model in which the real interest rate is the central bank's control variable. Hannsgen (2005) built a Minsky model with output depending on the *change* in a central-bank interest rate, based on the implications of maturity transformation by typical depository institutions. Also along horizontalist lines, Lima and Meirelles (2007) suggest a model with bank rate markup dynamics.

Another group of papers can be grouped together based on concerns related to sweeping changes in the financial structure and the distribution of income taking place since the 1970s. These include financial market inflation and increasing indebtedness of consumers. Hein (2012) applies the Kaleckian model, examining the stability of the household debt-to-capital ratio. Passarella (2011) constructs an SFC monetary-circuit account of falling financial soundness in an era of financialization. Bhaduri (2011) models an economy with rising debt and financial market inflation based on threshold principles, allowing for the kind of discontinuous phenomena that we attempt to model below. Skott (2013) looks at the effects of rising inequality on the stability of equity markets, using variables for portfolio allocation to equity, expected returns, and actual returns. Finally, Dafermos (2018) integrates a Kaleckian Minsky model into a higher-dimensional framework based on the national accounting identity that links the three main sectoral balances, à la Godley and Cripps (1983, 281–304).

Recent work by this author (e.g., Hannsgen 2012; Hannsgen and Young-Taft 2015) seeks to contribute to this literature. It features endogenous money (e.g., Robinson 1962, 45, 1970; Godley and Lavoie 2012, especially 127–28, 198–200; Rochon 2003), consistent with, for example, Lavoie (1986–87), Foley (2003), and Lima and Meirelles (2007). In such an approach, actors such as the government and banks have control over interest rates *at any given point* in time but rather naturally may change them over time in response to changes in economic conditions or psychological variables, but not simple excess demand for loanable funds or goods. These changes in interest rates may in turn influence the values of other variables. Moreover, of course, levels and growth rates of money are not under the *control* of policy authorities, as they are in the IS-LM framework adapted by Taylor and O'Connell (1985) (Minsky 1986, 129–33; Robinson 1970; Rochon 2003).

Following for example Dutt (2006) and Asada (2012), this paper allows *quantities* of (nominal-capital-stock-normalized) debt—rather than, say, putatively demand-led changes in interest rates—drive the tendency to financial fragility. With the addition of more state variables, we could also make bank rate markups or bond yields or both fluctuate, independently of the central bank rate (Hannsgen and Young-Taft 2015; Lima and Meirelles 2007). Thus, our endogenous-money approach in no way rules out dynamics in the interest rates that are most crucial for nonfinancial businesses and households.

Like for example, Dutt (2011) and Datta (2015), a series of papers by this author—written alone and with Young-Taft—has deployed intangible fragility variables, but it has sought to improve on previous efforts by conceptualizing fragility as a latent quantity determining the *probability* of a future crisis. As in Keen (1995), these models featured fiscal-policy reaction functions and variable income distribution.[3]

[2] See also Robinson (1970).

[3] On the other hand, they included independent investment-over-capital functions and variable capacity utilization, which would appear to us to align them more closely with the effective-demand vision of Keynes (1936), Robinson (1962), and Kalecki (1965).

Authors constructing dynamical Minsky models have often posited that growth and output are "debt burdened." For example, Asada's model (2012) featured debt–capacity utilization dynamics with feedback effects he summarized as follows

$$d \uparrow \rightarrow y \ \downarrow\downarrow$$

$$y \uparrow \rightarrow \ d \uparrow\uparrow$$

The double arrows indicate acceleration in rates of change. Higher debt reduces the rate of increase of output, and higher output increases the rate of increases of debt. Asada demonstrates the emergence of a limit cycle via a Hopf bifurcation. In another 2D approach with a quantity-of-debt variable, Taylor (2011, 195–97) posits a Goodwin-like Minsky cycle in the capital-stock growth rate and a debt ratio. His cycle starts with an exogenous downward jump in animal spirits.

Relatedly, one frequently used modeling strategy in *deterministic* models with financial fragility variables (e.g., Datta 2015, Dutt 2011) is to make borrowing, output, or the investment accelerator depend inversely on fragility. Deterministic debt-burdened or fragility-burdened models are characterized by self-limiting upturns—probably resulting in greater stability than Minsky saw in capitalist economies. In contrast, this paper's stochastic approach allows for a financial and output-employment boom followed by an abrupt bust, more in the spirit of Keynes's theory of crisis (1936, 315–324), as fleshed out by Minsky (e.g., 1977). Hence, our model will account in principle for irregular cycles in which borrowing-fueled booms develop, then end in sudden, hard-to-forecast, discontinuous events. Rosser (2019) has recently expressed the similar view that a Minsky crisis constitutes a "revenge of entropy."

The paradox-of-debt critique of Minsky emphasizes that the denominator of leverage measures can rise along with the numerator (e.g., Lavoie 1987; Seccareccia and Lavoie 2001). The model below features rising leverage in the sense of a rising ratio of private-sector debt to *capital.* Our presentation leaves open the nature of the private debt, meaning it could be household debt, as in Dutt (2006), Hein (2012), or Passarella (2012).

As in Dutt (2006), the model developed here uses a differential equation for a net *borrowing* ratio. Debt L modeled as the integral of net "financial risk-taking" b. In the model below, during a boom, (normalized) borrowing, capacity utilization, and (normalized) private debt all potentially increase. A crisis brings a sudden end to these self-reinforcing financial-real dynamics with a downward jump in all variables. We use the intuitive assumption that the instantaneous probability of a crisis is related positively to L and (optionally) negatively to capacity utilization u.

To revert for a moment to Asada's (y, d) notation for normalized output and debt plus our borrowing variable b, the deterministic part of the model below will combine the following unstable and zero-root feedback effects:

$$b \uparrow \rightarrow \ y \uparrow\uparrow$$

$$y \uparrow \rightarrow b \ \uparrow\uparrow$$

$$b\ high \rightarrow d \uparrow;\ and\ b\ low \rightarrow \ d \downarrow$$

Increases in the flow of net borrowing and aggregate demand reinforce one another. Also, net borrowing adds to debt via a stock-flow identity.

At times, especially in Section VII, we will also consider deterministic financial-stock effects that yield, among others, the three additional feedbacks:

$$d \uparrow \; \rightarrow \; \downarrow\downarrow b, \downarrow\downarrow y, \uparrow\uparrow d$$

The model of this paper is related to stock-flow consistent (SFC) models (Dos Santos 2005; Le Heron 2011; Lavoie and Godley 2012; Passarella 2012; Caverzasi and Godin 2015). Most current SFC models use a discrete time methodology and a relatively large number of variables. Hence, analysis of stability relies on simulations. In the SFC literature (e.g., Godley and Cripps 1983; Godley and Lavoie 2012; Le Heron 2008), changes in animal spirits and other psychological variables are often induced via assumed non-stochastic shifts that are routinely and perhaps confusingly referred to as *shocks* (Hannsgen 2013).

Our dynamic, stochastic approach allows for the simulation of histories with multiple jumps in which the probability of jumps depends on goods demand and debt.[4] Keynes (1936, 247) himself made it clear that, ultimately, his "psychological factors" were endogenous. Like the SFC approach, ours responds to an aspect of Robinson's point that an economy must be seen as evolving in historical time, rather than moving toward equilibrium in "logical time" or always remaining in equilibrium (Robinson 1962, 23–39). Crises induce path-dependent change. The endogeneity of the arrival rate of crises to a financial ratio is one crucial way that the model differs sharply from neoclassical stochastic approaches that attribute macroeconomic fluctuations to random movement.

Justifying this new work: In the previous work mentioned above by this author, the Minsky block was part of a larger Post Keynesian model drawing from Kaldor (1940) and Kalecki (1965) that focused on the use of public spending with chartal (fiat) money to stabilize fluctuations in capacity utilization (Hannsgen 2012; Hannsgen and Young-Taft 2015). The Minsky component superimposed a financial cycle on Kaldorian business cycle dynamics tempered by countercyclical fiscal policy, with the added complication of "Radical"-stagnationist[5] (Taylor 1985, 386) distribution-output dynamics. So far, Hannsgen (2014), a revised version of Hannsgen (2012), is the only published result of these efforts, and the Minsky block was excised from that article for the sake of brevity and concrete results. Since the Poisson Minsky model is not intrinsically connected to the larger framework in that paper, it can indeed be stated in isolated form.

[4] The literature proposing Minsky models has developed in a number of other directions. Charles (2008) endogenizes the corporate retention rate. Ryoo (2013) offered an alternative way of resolving the seeming dilemma of the paradox of debt versus Minsky. Charpe et al. (2011) offered a variety of mostly neoclassical synthesis approaches using continuous time dynamic models. Minsky's onetime coauthor Ferri has demonstrated (e.g., 2011) the utility of a regime-switching stochastic model for modeling Minskyan instability. The model developed in this paper features point-in-time crises, like those in Nishi (2012) and Bhaduri (2011, 1005–07), but these latter are based on *deterministic* threshold phenomena. Finally, I have not addressed the literature developing models with large numbers of heterogeneous agents.

[5] The term Radical in the Taylor's terminology stems from the use of a countercyclical markup or profit share in models by many members of the Radical Political Economy school and the fact that the theory of distribution most closely associated with the modern Post Keynesian school in contrast assumes a procyclical markup. Kalecki (1965) and Goodwin (1967) adhered to the "Radical" theory long before the emergence of the Radical school, and it has gained other supporters since then.

Contents of paper: In this paper, I formulate just the financial instability mechanism in these earlier papers in a dynamical model with three state variables: b (financial risk-taking or borrowing), u (capacity utilization), and L (accumulated debt). Additionally, the Poisson model uses a time variable t, the endogenous arrival rate parameter λ, and an endogenous parameter in the distribution from which shocks are drawn. The resulting model accounts for endogenous instability but omits Minsky's thwarting systems, such as "Big Government" and "Lender of Last Resort" (e.g., Minsky 1986, 13–67; Ferri and Minsky 1992; Dafermos 2018). Also, as in most "Minsky models," distribution and inequality (e.g., Skott 2012) are not separately modeled. The model can be expressed in terms of

(1) three differential equations,
(2) a compound stable-Poisson stochastic process, and the functions that give the endogenous values of the parameters of the process.

Thus, this paper will present in a careful way the most basic part of the formalization of Minskyan instability initially suggested in Hannsgen (2012, 2013) and Hannsgen and Young-Taft (2015).

Detailed organization of the remainder of the paper: Section II presents the deterministic dynamical system, along with analytical results in the form of symbolic expressions for the roots of the linear part of a simple version of the model. I show how this part of the model generates endogenous upward or downward instability. Section III presents the model used to generate stochastic jumps at an endogenous rate that increases with financial fragility. Section IV presents a more elegant statement of the model. Section V presents a model of nonlinear self-correcting forces that can bound the deterministic system. Section VI presents a more–SFC version of the model. This move forces the use of effects that we have set equal to zero in the simplified model of Section II and adds an additional effect. Computing the eigenvalues of the linear, deterministic part of this more-SFC system requires the use of a numerical example rather than analytical methods, though stability conditions are stated. Section VII is a conclusion that summarizes the methodology and results, suggests limitations of the present study, and argues that the model captures key elements of Minsky's theory of financial instability. Section VIII is an appendix covering (a) computation of the analytical derivatives of the debt integrals for both models, (b) computation of the (meaningless) stationary equilibrium for the model of Section II, (c) computation of the eigenvalues, and (d) analytical stability conditions for the more-SFC model of Section VI.

II. The Differential Equations in the Continuous-time Variables: b, u, L: Equilibrium and Qualitative Dynamics for a Simplified Version

We seek a system that will contain three dynamically endogenous variables, b, u, and L. Our notation will leave dependence on time implicit.

b: Let b be the rate of net accumulation of debt relative to the capital stock. This variable can presumably take on both negative and positive values.

u: Let u equal capacity utilization, defined so that it takes on values between zero and one. That is, let

$$u = Y/\nu K$$

where Y is output per unit of time in physical units, K is the capital stock in physical units, and the parameter ν is defined as output per employed physical unit of capital per unit of time.

L: Our measure of total systemic risk, L, given that the last crisis occurred at time t_0 is generally

$$L = \int_{t_0}^{t} f_1\big(b(s)\big)\, ds, \qquad f_1' > 0$$

Generally, b adds to the stock of risky debt in some potentially nonlinear way through the function f_1. For the sake of simplicity, we specialize to a simple linear specification for f_1

$$L = \int_{t_0}^{t} b(s)\, ds \quad (1)$$

We will think of b and L as indexes expressed in relation to capital.

Next, I posit a probability model of crisis in the form of a Poisson process, a stochastic process that jumps when an event occurs (Ross 1997, 235–287; Walde 2011, 225–38). The parameter λ in the process will depend on L and u through an intensity function as follows

$\lambda = f_2(L, u),\ \partial f_2/\partial L > 0,\ \partial f_2/\partial u\ < 0$

The use of endogenous λ means that this Poisson process is nonhomogeneous (Ross 1997, 277–281). The variable λ is a latent measure of fragility that can rise over time, while allowing some of the effects of a boom in borrowing to occur with a delay. In neoclassical Schumpeterian endogenous growth theory, λ is sometimes an increasing function of labor devoted to research and development, yielding a model of the rate of innovation (Aghion and Howitt 1997, 53–83).

For simplicity, we will assume that the effects of u are small enough to ignore, i.e.,

$$\frac{\partial f_2}{\partial u} = 0$$

allowing accumulated risk L to be the sole variable determining probability of crisis.

In the homogeneous type of Poisson process, the number of crises in an interval of time of length t is distributed as Poisson with mean λt, and the time between crises is distributed as an exponential random variable with mean 1/λ. The corresponding distributions for a nonhomogeneous Poisson process depend on a mean value function obtained by integrating over a particular time interval to obtain the probability of a crisis in the interval (Ross 1997, 277–78).

We assume that the rate of change of the addition to financial risk is increasing in capacity utilization u (an effect of general economic euphoria) and decreasing in accumulated risk L. Further, a function in which higher L reduces new financial risk-taking b may make sense. For example, lenders may restrict lending as the private-sector debt burden rises because of concerns about default, a stock-flow consistent (Godley and Lavoie 2012) effect.

$$\frac{db}{dt} = \alpha_{b0} + \alpha_{bu}u - \alpha_{bL}L \quad (2)$$

Next, we posit that capacity utilization u follows the law of motion

$$\frac{du}{dt} = \alpha_{u0} + \alpha_{ub} b + \alpha_{uu} u - \alpha_{uL} L \quad (3)$$

$$\alpha_{uu} < 1$$

This variable is driven upward by borrowing connected with spending. We assume a direct own-effect for the variable u—upward motion is self-reinforcing.

Differentiating the integral for L (eq. 1) by t, one obtains

$$\frac{dL}{dt} = b(t) \quad (4)$$

We then have a system made up of the ordinary differential equations (2), (3), and (4) in the variables b, u, and L.

A previous use of a private debt function in a similar-sized dynamical macro model would be Chiarella, Flaschel, and Semmler (1999, 124). In that neoclassical model, debt accumulation is determined through a household intertemporal optimization process, while we will use the non-optimizing formulation above, in which u and L appear in an equation for db/dt.

In addition, we can keep an initial condition from our integral equation that is lost in differentiation.

$$L_{t_0} = L_0 \quad (5)$$

In matrix form, the system in b (net borrowing), u (capacity utilization), and L (debt) is:

$$\begin{bmatrix} db/dt \\ du/dt \\ dL/dt \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{bu} & -\alpha_{bL} \\ \alpha_{ub} & \alpha_{uu} & -\alpha_{uL} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ u \\ L \end{bmatrix} + \begin{bmatrix} \alpha_{b0} \\ \alpha_{u0} \\ 0 \end{bmatrix} \quad (6)$$

The matrix is nonsingular, so the mathematical equilibrium of this nonhomogeneous system exists and is unique. See part (b) of the appendix (Section VIII) for an analytical solution for the more-complicated model of Section VI. We discuss possible meaningful equilibria from the perspective of economics in more detail below.

Feedback Loops: The 3×3 coefficient matrix in eq. (6) shows that the feedback loops in this system are:

1) b, u dual self-reinforcing dynamics, e.g., $b \uparrow \rightarrow u \ \uparrow\uparrow \rightarrow \ b \ \uparrow\uparrow\uparrow$
2) u self-reinforcing dynamics: $\uparrow u \rightarrow u \uparrow\uparrow \rightarrow u \uparrow\uparrow\uparrow$
3) L, b self-correcting dynamics: $b > 0 \rightarrow L \uparrow \rightarrow b \downarrow\downarrow$
4) L, u, b self-correcting dynamics: $b > 0 \rightarrow L \uparrow \rightarrow u \downarrow\downarrow \rightarrow b \downarrow\downarrow\downarrow$

Double up or down arrows indicate acceleration, while triple up or down arrows denote acceleration of rates of change.

Stability Analysis: Eigenvalue analysis for the system in symbols (eq. 6) can begin with the corresponding homogeneous system obtained by omitting the vector of constants. The trace of the coefficient matrix is greater than zero, violating one of the three Routh-Hurwitz conditions for a system of this size (Gandolfo 1997, 251–52). Hence, this linear system is not stable.

It would be helpful to have explicit expressions for the eigenvalues. In the interest of computational simplicity, I tried assuming α_{bL}= 0. A program to find the eigenvalues produced a very complicated set of expressions. I then tried making the additional simplifying assumption that α_{uL} = 0. These assumptions unfortunately remove the effects of past borrowing on current borrowing and on output via aggregate demand. In the resulting model, the only self-correcting force will be the downward jumps to be developed in the next section.

We can think of the shock as a *probabilistic* stock-flow mechanism, in that it takes into account medium-term impacts of the accumulation of financial risk. Moreover, we will take up a version with the omitted deterministic SFC effects in Section IV using a numerical example. Finally, in Section VII, we will find an analytical stability condition in symbols for a more-SFC model, though we will again refrain from reporting complicated and unhelpful eigenvalue expressions.

The matrix of coefficients for the simplified model is

$$\begin{bmatrix} 0 & \alpha_{bu} & 0 \\ \alpha_{ub} & \alpha_{uu} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using this matrix, the revised system of equations is:

$$\begin{bmatrix} db/dt \\ du/dt \\ dL/dt \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{bu} & 0 \\ \alpha_{ub} & \alpha_{uu} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ u \\ L \end{bmatrix} + \begin{bmatrix} \alpha_{b0} \\ \alpha_{u0} \\ 0 \end{bmatrix} \quad (6')$$

Feedback Loops in Simplified Model (eq. 6'): Setting debt feedback coefficients equal to zero as in eq. (6') removes the third and fourth feedback loops in the list above. We replace those feedbacks with this one-way causal mechanism:

- $b > 0 \rightarrow L \uparrow$

Thus, all remaining feedbacks are positive. Also, these feedback mechanisms can work in the opposite direction, causing all variables to fall continuously. As we will see, depending on initial conditions, these feedbacks may not work in this self-reinforcing way in the short to medium run.

Stability Analysis for the Simplified Model (eq. 6'): The column of zeros in the matrix in (6') implies a zero root. In fact, MATLAB® computed the following three eigenvalues for the simplified coefficient matrix:

$$\left\{0, \frac{\alpha_{uu}}{2} + \left(\frac{\sqrt{\alpha_{uu}^2 + 4\alpha_{bu}\alpha_{ub}}}{2}\right), \frac{\alpha_{uu}}{2} - \left(\frac{\sqrt{\alpha_{uu}^2 + 4\alpha_{bu}\alpha_{ub}}}{2}\right)\right\}$$

See the part (c) of the Appendix (Section VIII) for the code and output.

Moreover, an expression for stationary equilibrium in eq. (6') indicates no unique solution, since for example, our equations do not restrict L. In fact, the zero root means that the solution for the simplified model will give a *center subspace*. (See, for example, Hirsch and Smale 1974, 109–143.)

Looking at the homogeneous version of eq (6'), we find that borrowing b is zero at equilibrium. (Nonzero b implies dL/dt $\neq$ 0 in the third row.)

$$\begin{bmatrix} db/dt \\ du/dt \\ dL/dt \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{bu} & 0 \\ \alpha_{ub} & \alpha_{uu} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ u \\ L \end{bmatrix} \quad (7)$$

Furthermore, as we have seen above, equilibrium is not stable. This system will not approach equilibrium, nor is the latter Minskyan, as, with b = 0, it lacks a tendency toward financial fragility.

From our theoretical perspective, we are more interested in a weaker equilibrium concept, allowing us to find parts of the state space in which normalized debt L rises over time. Formally, we would be interested in solutions to equation (7) with the following characteristics

Equilibrium Concept (Minsky moving equilibrium): $\frac{db}{dt} = 0;\ \frac{du}{dt} = 0;\ \frac{dL}{dt} > 0\ in\ the\ homogeneous\ system\ (eq. 7)$

Applying this concept will yield a solution of the following form:

$$b = b^* > 0, \qquad u = u^* \in [0,1], \qquad L = Ce^{bt}$$

I assume an initial condition of t = 0. Such a moving equilibrium will be a closer to a match to a real-world economy in which the stock of risky debt has a tendency to rise relative to some normalizing variable—Minsky's endogenous emergence of instability (e.g., 1977, 66–67; 1986, 213).

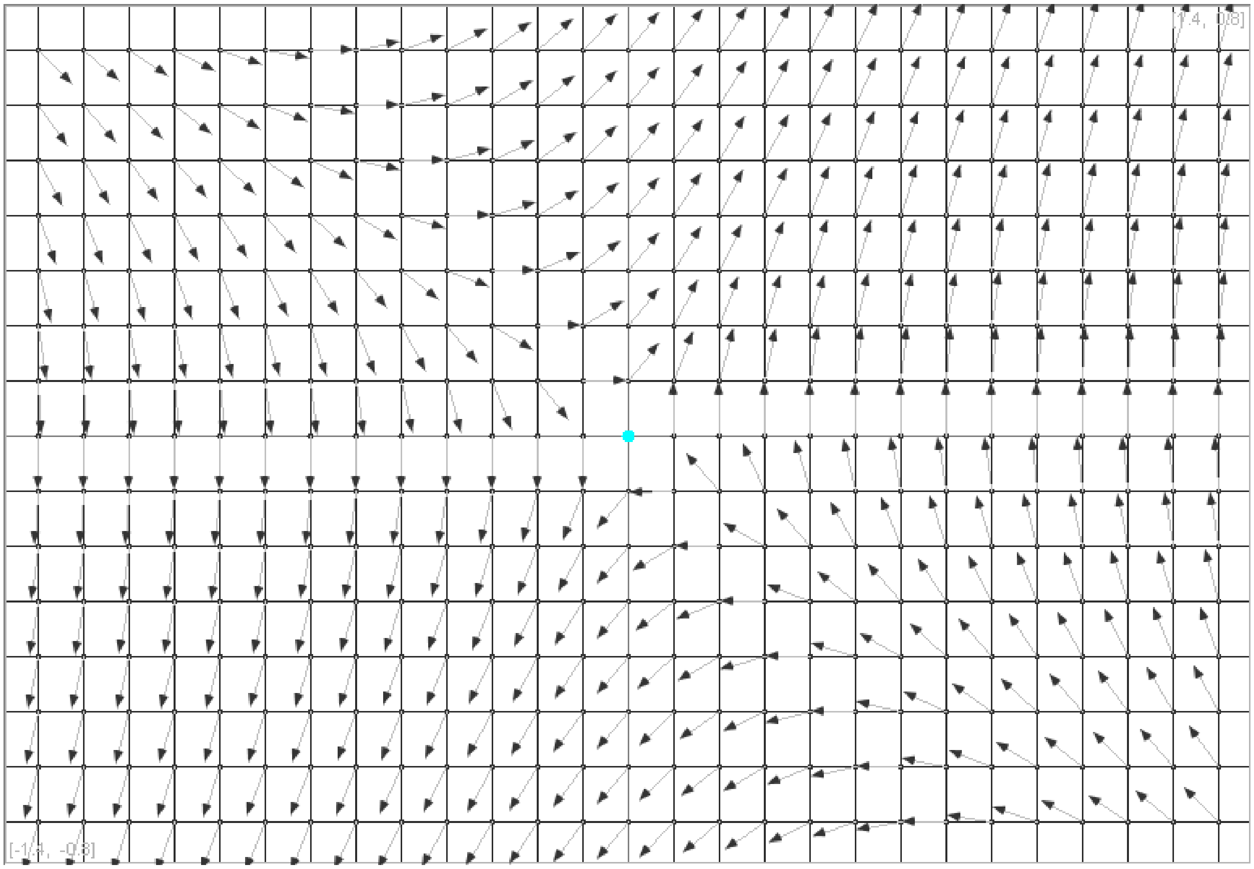
The stability of the moving equilibrium can be analyzed as follows. Suppose we transform to coordinates b, u, and L_τ= $C.exp(b^*t)$. We would then have a space which L was approximately motionless for b near the moving equilibrium. Looking at 2D dynamics, the coefficient matrix for the reduced system is:

$$\begin{bmatrix} 0 & \alpha_{bu} \\ \alpha_{ub} & \alpha_{uu} \end{bmatrix}$$

The trace is positive and the determinant negative, implying roots of opposite sign. The positive and negative roots imply unstable saddle-point-type dynamics in a b, u subsystem. The stable subspace in the reduced system is only one-dimensional, with the system ultimately pushing away from the moving Minsky equilibrium elsewhere. No tendency exists for the rates of borrowing or capacity utilization to stabilize. The phase diagram (made by PHASER 3.0) below shows b on the abscissa and u on the ordinate.

CONTINUED BELOW

FIGURE 1: SCHEMATIC PHASE DIAGRAM FOR THE b, u SUBSYSTEM



To ascertain the qualitative dynamics, I have used a system with ones in the nonzero entries of the coefficient matrix and zeroes for the constants. The $\dot{b} = 0$ locus is horizontal, while the $\dot{u} = 0$ locus is downward sloping. Given positive equilibrium b, the point of intersection of the two loci is a Minsky moving equilibrium, since L—normalized debt—is increasing there, while b and u—borrowing and capacity utilization—are constant. The 2D figure shows that regardless of initial conditions, b and u eventually rise or fall along any trajectory, except on the usual measure-zero stable arms of the saddle.

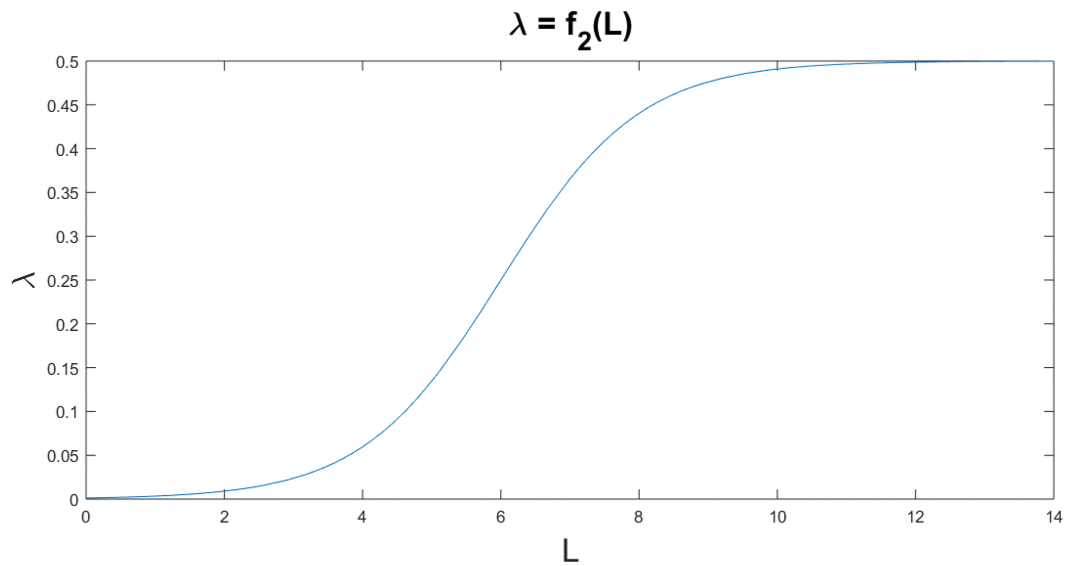
We return to untransformed 3D space. L once again rises continuously. The zero root implies a 2D center subspace, and there exists a 2D stable subspace, corresponding to the 1D stable arm of the 2D system that we just considered (Hirsch and Smale 1974, 109–143). The 2D center subspace partitions the meaningful (3D) state space into invariant subspaces characterized by eventual upward and downward instability, respectively. The constants in the equations obviously matter for the relative volume of the upwardly and downwardly stable regions.

Summary: We now possess a simple system that in the long run generates increasing borrowing and capacity utilization or falling values of both of those variables, with no tendency to converge to a steady state. At the least, borrowing or capacity utilization are moving away from the moving equilibrium. The aftermath of a crisis brings a gradual, endogenous acceleration of risky borrowing. In Section III, we will add stochastic jumps that represent the next element from Minsky's theory—crises in which optimism, financial risk-taking, and outstanding debt suddenly collapse.

III. The Crisis-Shock: A Stochastic Discrete Jump on Crisis Dates

We next seek to model how a sudden change in financial conventions or expectations can drive the emergence of crisis, interrupting the self-reinforcing cycle of increasing confidence and capacity utilization. It may be that all know debt ratios are going to eventually cause a crisis, but because no one knows the timing of this future event, debt can continue to increase—a financial scenario familiar to readers of Keynes (1936) and Minsky (e.g., 1977, 1986).

FIGURE 2: POISSON ARRIVAL-RATE PARAMETER λ AS A FUNCTION OF L, NORMALIZED LIABILITIES



We make the parameter governing the frequency of crises endogenous as follows

$$\lambda = f_2(L)$$

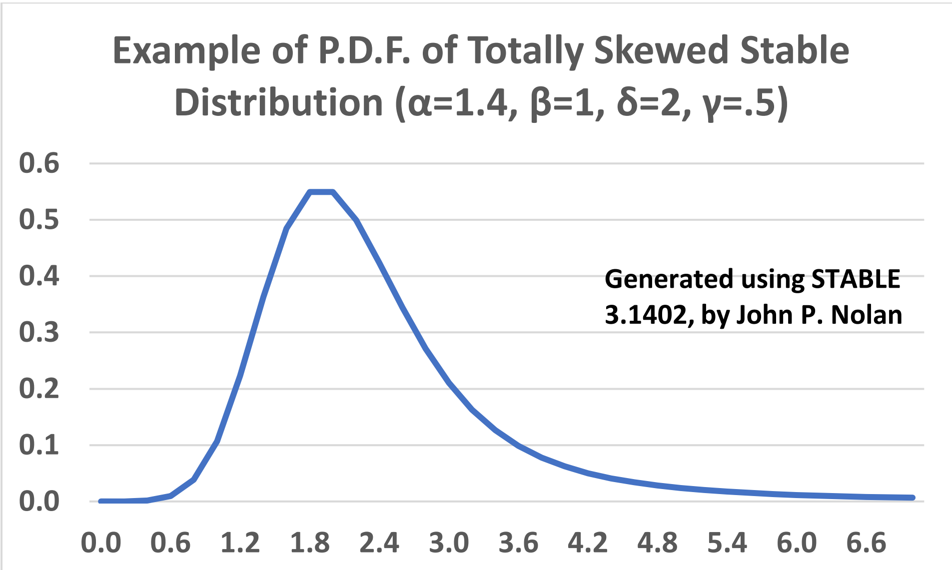
where f is a logistic function. With this specification, f has a lower bound at zero and a positive upper bound $\bar{\lambda}$. In this formulation, λ, the frequency of crises reflects the stock of debt, a form of stock-flow consistency (Godley and Cripps 1983; Godley and Lavoie 2012) that we can call *probabilistic stock-flow consistency.*

Suppose an arrival date occurs at time t_0 in a realization of the Poisson process driven by λ. A shock $\varepsilon_T > 0$ is drawn from a distribution, F. We use a non-Gaussian stable distribution. These distributions and their properties were discovered by Paul Lévy in the 1930s and are the subject of a book by Nolan (2020). They were applied in physics and financial economics as part of the complexity revolution in the late 20th Century, with the leading scientific figure being Benoît Mandelbrot, but were suppressed in mainstream macroeconomics beginning in the 1970s (Mirowski 1990; Sent 1998). This class of distributions generalizes the normal distribution to allow for heavy tails and skew. In one parameterization, a stable distribution possesses the parameters

$$\alpha, \beta, \delta, \gamma$$

with restrictions 0 < α ≤ 2, −1 ≤ β ≤ 1, and γ > 0. We will choose a parameter vector satisfying 1 < α < 2 and β = 1. See the example probability density function below.

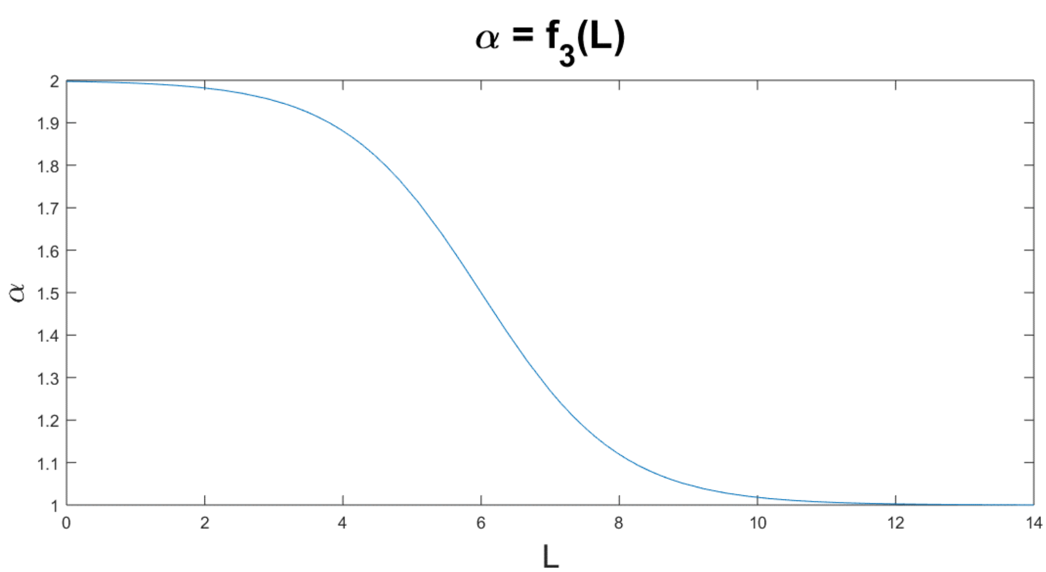
FIGURE 3: PDF OF TOTALLY SKEWED, INFINITE-VARIANCE STABLE DISTRIBUTION



The choice of α < 2 ensures we have an infinite-variance case rather than the normal distribution. On the other hand, our assumption that α > 1 ensures that the mean exists (is finite). Finally, with β = 1, such a distribution is totally skewed to the right, meaning that its support is bounded to the left (Fofack and Nolan 1999). A stable distribution of this type has asymptotic power law behavior as $\varepsilon \to \infty$. Our framework would also allow us to make the thickness of the tail a function of the business cycle variable u or accumulated risk L or both, so that

$$\alpha = f_3(u, L)$$

FIGURE 4: STABLE TAIL-THICKNESS PARAMETER α AS A FUNCTION OF L, NORMALIZED LIABILTIES



We want to specify a way that our heavy-tailed random variate ε changes the variables in the model, b, u, and L. Suppose we are looking at the trajectory of a given simulation with initial conditions b_0, u_0, L_0. A shock causes a discontinuous jump at time T of impact

$$\Delta \begin{bmatrix} b \\ u \\ L \end{bmatrix} = - \begin{bmatrix} b * (1 - exp\,(-\varepsilon^{-1})) \\ u * k_1 * (1 - exp\,(-k_1 \varepsilon^{-1})) \\ L * (1 - exp\,(-k_2 \varepsilon^{-1})) \end{bmatrix}$$

where k_1 and k_2 are constants. This specification implies that a crisis causes instantaneous downward jumps in borrowing b, capacity utilization u, and debt L. Of course, Minsky (1977, 1986) held that crises invariably led to a temporary return of financial conservatism, as well write-downs of risky debt.

We have constructed the impacts to avoid situations in which the shocks carry one or more state variables beyond the intervals in which they have economic meaning. The continuous 3D system of the previous section restarts at t = T with new initial conditions, resulting in a right-continuous trajectory at that point. If the Poisson process generates no event in some interval $[t_0, t_1]$, these discrete jump variables remain equal to zero. The mixture of continuity and jumps characterizes what is known as a *cádlág* process (Walde 2011, 261).

We show how to express the complete model in an elegant way in the next section.

Summary: We now have a combination of a continuous system in three variables b, u, and L; the discrete jumps in the same variables Δb, Δu, and ΔL; and the endogenous parameters λ and α. In the region characterized by upward movement in all variables, the jumps create an effect of irregular cycles. A large enough jump would carry the economy into the region that is characterized by falling values of all variables—a catastrophic event leading to irreversible change, after which further jumps only hasten downward motion in state space. In the regions in which b and u move in opposite directions, catastrophic drops can also occur, leading abruptly to continuous downward motion in b *and* u.

IV. A More Elegant Statement: Three D.E.s and the Stable-Poisson Process

Now that we have set forth the model, we can find a more mathematically elegant formulation.

We will make use of the integral representation for the compound stable-Poisson stochastic process (eq. 8) of the previous section

$$q(t) - q(t_0) = \int_{t_0}^{t} p(s)\, ds$$

The increment dq of this process equals zero when no crisis occurs, while

$$dq \sim F(f_3(u, L), 1, \delta, \gamma)$$

at jump points, where F is a totally skewed, infinite-variance stable distributione, and f_3 gives the value of the tail-thickness parameter α, as described in the previous section.

Summary: Adding the new component to (7), the model can be stated in the form of three differential equations in the three endogenous variables and the increment of the stochastic process, as explained in Walde (2011, 236)

$$\begin{bmatrix} db \\ du \\ dL \end{bmatrix} = \begin{bmatrix} dt & dt & dt \end{bmatrix} \begin{bmatrix} 0 & \alpha_{bu} & 0 \\ \alpha_{ub} & \alpha_{uu} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ u \\ L \end{bmatrix} - \begin{bmatrix} b * (1 - \exp(-dq^{-1})) \\ u * k_1 * (1 - \exp(-k_1 dq^{-1})) \\ b * (1 - \exp(-k_2 dq^{-1})) \end{bmatrix} \quad (8)$$

together with the functions f_2 and f_3 giving the values of λ and α and the remaining parameters of the stable distribution F, γ and δ. Once initial conditions are known, a constant vector can be added.

V. Possible Nonlinearities and Boundedness

We comment now on some properties of the state space of our continuous-time model that we can develop by arguing on for the plausibility of a positively invariant, open, bounded region of this space. We are led to impose bounds by the economics of the system. These bounds become important if the system is in a boom phase with rising b, u, and L and by luck avoids crisis-jumps, which usually mark the end of a boom. Formally, we could augment system (8) with functions that ensure that behavior is bounded as the economy approaches limits of plausible or defined behavior (Flaschel, Franke, and Semmler 1999, 86–87).

We then consider the nonlinear system

$$\begin{bmatrix} db \\ du \\ dL \end{bmatrix} = \begin{bmatrix} dt & dt & dt \end{bmatrix} \begin{bmatrix} 0 & \alpha_{bu} & 0 \\ \alpha_{ub} & \alpha_{uu} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b \\ u \\ L \end{bmatrix} - \begin{bmatrix} b * (1 - \exp(-dq^{-1})) \\ u * k_1 * (1 - \exp(-k_1 dq^{-1})) \\ b * (1 - \exp(-k_2 dq^{-1})) \end{bmatrix} + \boldsymbol{g}(b, u, L) \quad (9)$$

where we have simply added the term

$$\boldsymbol{g}(b, u,\ L) = \begin{bmatrix} g_1(b,u,L) \\ g_2(b,u,L) \\ g_3(b,u,L) \end{bmatrix}, where\ g = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} in\ a\ region\ near\ equlibrium, and\ \frac{\partial g_i}{\partial b}, \frac{\partial g_i}{\partial u}, \frac{\partial g_i}{\partial L} \leq 0, i = 1,2,3$$

The added term pushes the economy away from the boundaries as b, u, or L reach the extremes of the economically meaningful state space. These nonlinear feedbacks correct the extremely unstable behavior of the deterministic part of our model, in which no negative feedback loops exist even for very high or low values of capacity utilization, borrowing, or loans. This need becomes relevant in “lucky” realizations, in which a large amount of time elapses between crises, or jumps are small, or both.

Additional thoughts about reasons to expect inward-pushing forces that push trajectories away from the outer boundary might include: (1) lenders are likely to impose limits on b in practice since net accumulation of debt makes no sense once we are substantially above or below the range of observed values, given an existing institutional framework, including prevailing lending practices; (2) typical Steindlian forces are likely to bound u as we reach extreme levels of that variable. Namely, extremes of u are corrected by stabilizing changes in investment, pushing the economy toward moderate levels of u (Flaschel, Franke, and Semmler 1999); (3) for high levels of L, movement toward greater u and b may be eventually reversed, owing to Fisher-Kalecki-Keynes-Tobin debt-burden effects, which, as explained above, we have assumed away in constructing our formal analysis of the roots of this system.

Therefore, L is bounded in part by changes in the behavior of b and u, which tend to reinforce one another, setting up a reverse dynamic and leading to falling L—actual declines in net indebtedness through credit rationing or borrower efforts to repay or both. Falling L behavior might occur in part through the writing off of bad debt rather than declines in lending minus repayments. Write-offs of the stock of debt would be modelled as self-correcting, nonlinear own-effects in the equation for L at very high values of this variable.[6] We introduce some negative feedback effects in the linear portion of the system in the next section.

VI. A Minsky Model with Two Deterministic Stock Effects: An Unstable Case and General Stability Conditions

We will next suggest how to incorporate some deterministic SFC effects into the model. This will require dropping the simplifying conditions of Section II through V. While we now lose our earlier tractable symbolic expressions for the roots of the model, we can state a set of necessary and sufficient stability conditions. In fact, in part (d) of the Appendix (Section VIII), I show that generally the linear part of the new system is stable under a set of three necessary and sufficient conditions.

Intuitively, our more-SFC system will not under all conditions have self-reinforcing debt-output dynamics that dominate until a crisis occurs. Stability of the linear, deterministic part will depend on the parameter values. The system will be less inherently unstable because high normalized debt levels L slow the rates of increase of borrowing b and capacity utilization u.

For this simple model of Minsky's theory of financial instability, it seems reasonable to select a case for which local stability is provided *only* by financial crises that abruptly bring booms to an end. Hence, we will compute the roots for a numerical example in which the linear part of the system is *not* stable, as in the simplified model considered above. As before, the ad hoc nonlinearities introduced in Section V will bound the system globally, preventing normalized debt and other variables from attaining values where they lack economic meaning or plausibility, regardless of the realizations of the Poisson process.

The Revised Model: We now define financial variables in a more tangible way. Let b be new borrowing and L be accumulated debt. Let each variable be the ratio of its nominal counterpart to the real capital stock K times the fixed goods price p*. Using the variables B for nominal borrowing and $L^{\{N\}}$ for the nominal stock of debt:

$$b = \frac{B}{Kp}$$

$$L = L^{\{N\}}/Kp$$

$$p = p^*$$

[6] An invariant region generally contains limit sets, including an equilibrium. In principle, the qualitative possibilities for such a region in three dimensions are known to be numerous and possibly complex, including chaotic attractors, as well as limit cycles.

We assume that we can ignore capital gains on financial assets. Using a revised f_1, the definition of the stock variable L becomes

$$L = \int_{t_0}^{t} b - \rho L \, ds \quad (1')$$

Here, I use 0 < ρ < 1 for the repayment rate, and b now represents *gross* borrowing relative to capital.

The new differential equation for L is

$$dL/dt = b - \rho L \; (4')$$

See part (a) of the Appendix (Section VIII) for the computation of this time derivative.

Let i equal the interest rate. We can now add an "SFC" term in L for the debt-service burden to the equations for b

$$\frac{db}{dt} = \alpha_{b0} + \alpha_{bu} u - \alpha_{bL}(i + \rho)L \quad (2')$$

and u

$$\frac{du}{dt} = \alpha_{u0} + \alpha_{ub} b + \alpha_{uu} u - \alpha_{uL}(i + \rho)L \quad (3')$$

Use of such a term is justified in part by cash-flow effects for firms and households that do not have ready access to credit markets and cannot easily increase mark-ups. The model still lacks full SFC accounting, which would require a much larger model and a simulation methodology best suited for the use of difference equations.

Our coefficient matrix for the linear terms in the system made up of (2'), (3'), and (4') is now

$$\begin{bmatrix} 0 & \alpha_{bu} & -\alpha_{bL}(i+\rho) \\ \alpha_{ub} & \alpha_{uu} & -\alpha_{uL}(i+\rho) \\ 1 & 0 & -\rho \end{bmatrix}$$

Right away we see that the system will be unstable for some values of the parameters. For a given value of the (3, 3) element, we can make the trace positive by making α_{uu} sufficiently large. The equilibrium of the linear system is stated in Part (b) of Section VIII. Part (d) of Section VIII contains formal stability conditions.

New Feedback Loops: We have now returned to the four feedback loops of the original (nonsimplified) analytical model, plus one:

(1) higher debt repayment and interest payments again lead to lower borrowing and capacity utilization as in the model in eq. (6).
(2) The added effect is an own-effect of debt on increases in debt through an interest burden effect in the (3, 3) element of the coefficient matrix.

$$L \uparrow \rightarrow L \uparrow\uparrow \rightarrow b, u \downarrow\downarrow$$

We have stated the directions of the effects for the case of upward changes in L. Of course, downward changes in L lead to effects in the opposite direction.

Numerical Example: For our numerical example, we will try the following coefficient matrix

$$\begin{bmatrix} 0 & .8 & -.15 \\ .8 & .9 & -.1 \\ 1 & 0 & -.1 \end{bmatrix}$$

Here I have used $\alpha_{bL} = .5, \alpha_{uL} = \frac{2}{15}, i = .2, \rho = .1$. The rest of the parameter values are obvious.

Results for Numerical Example: MATLAB® computed the following eigenvalues: 1.3114, –0.2557+0.1592i, –0.2557–0.1592i . Since at least one root has positive real part, the system is unstable. Since two of the roots are a complex conjugate pair, the linear part of this system is characterized in part by spiraling motion (rotation). As before, our nonlinear ad hoc addition can bound the deterministic system. Finally, as in the less-SFC, analytically tractable example, the stochastic stable-Poisson jump model will generate downward jumps in the variables at an endogenous rate.

Summary: The use of a more-complete model with SFC features illustrates the possibility of a different *type* of instability in the linear, deterministic part of the system. The irregular, repeated stochastic jumps tame upward instability to some extent. The imposed nonlinear, deterministic bounds (Section V) also limit trajectories in any case. For some parts of the parameter space, the revised linear system is stable. (See Section VIII (d).) As in the papers by, e.g., Dutt (2006) (see Section I), deterministic debt feedback seems to limit the scope for modeling Minskyan upward instability—the endogenous emergence of financial fragility.

VII. Conclusion: The Simplified-Analytical and Stock-Flow Models, This Paper's Findings, and How All Correspond to Minsky's Theory

The Model: Summing up the model itself, the general model in symbols has three variables: (1) normalized flow financial behavior, b; (2) a demand-driven capacity utilization variable, u; and (3) accumulated financial risk, L. We posited a system of three linear differential equations in these variables. The linear behavioral equations embody self-reinforcing dynamics in b and u, representing endogenous upward and downward instability, as well as a mechanical relationship between b and L, in which L was the integral of some function of b. To generate jumps whose probability varied with financial fragility, we added (1) a Poisson process with an arrival rate λ that was a function of L and (2) a shock with an asymmetric, non-Gaussian, stable distribution whose tail-heaviness parameter α was a function of L. Finally, this narrative demonstrated how to augment the system with behaviorally motivated nonlinear terms in each of the equations. These negative-feedback terms ensure the global boundedness of the deterministic, continuous-time system in the absence of Poisson jumps. These nonlinear terms were assumed to equal zero toward the interior of the economically meaningful state space.

Methodology: In this paper, we have considered two approaches to finding the qualitative dynamics of the model. First, we considered a clearly non-SFC analytical model. We sought to compute the roots of the system. To obtain a more analytically tractable case in symbols, we simplified by eliminating two posited effects in the linear part of the model, setting two coefficients equal to zero. Second, we considered a closely related model with additional SFC, behavioral effects. This model assumed constant repayment and interest rates for private-sector debt and allowed total debt-service payments to have negative feedback effects on the rates of change of b and u.

Results: We analytically computed expressions for the roots of a simplified version of the linear, deterministic part of the model. We found that this part of the model had real roots of opposite sign along with a zero root. The state space is made up of a two-dimensional invariant center manifold, a two-dimensional stable manifold, and saddle-point-unstable, three-dimensional regions. A phase diagram illustrated the saddle-point dynamics of b and u in a system transformed so that L remained approximately constant.

Given these dynamics, particular solutions will eventually generate upward or downward drift in all variables, leading to a monotonically increasing endogenous arrival rate of crisis. We called the upwardly unstable case a *Minsky moving equilibrium*. Also, owing to endogenous α, the right tail in the stable distribution from which jumps are drawn becomes heavier. This movement *increases* the probability that the next downward jump will be an extreme event.

The paper next considered the possibility of analyzing a more-SFC version of the same model. In this case we add interest payments entailed by the stock of debt. All stocks and flows become ratios to the (constant price times the) capital stock. We limit debt effects to effects of repayment and interest commitments, rather than making them unrestricted multiples of the debt variable. In this model the normalized stock of debt L provides negative feedback in the behavioral equations for b and u via these flow commitments. Our conclusions were slightly modified. Specifically, the linear part of the new system was stable for some parameter values. For a specific unstable case, the linear system in b, u, and L was characterized by some rotation, as evidenced by a pair of complex roots. The zero root had disappeared.

Limitations of this Study: Throughout, we left out many factors in order to reach a minimal model of Minsky's idea in three state variables. We have neglected possibly endogenous labor supply and productivity growth. We have not modeled public or external-sector behavior. We have assumed that wages, prices, interest rates and repayment rates were constant. We have included only one type of generic private financial liability, neglecting for example firm equity. We have not looked in detail at the sources of demand for the products of the private sector. We have not modeled the net income of any sector or group of households, including, for example, groups that are especially vulnerable to predatory lending. We have not considered intrahousehold distribution or behavior. We have not modeled the capital stock, but instead left it as an implicit normalizing variable in u and a factor in the denominators of b and L. We have not considered markets for financial assets. Finally, we have not estimated the crisis-probability-and-magnitude model.

The Model of this paper and Minsky's Theory of Financial Instability (e.g., Minsky 1977, 1986). The key properties of the economy and Minsky's theory captured by this model are (1) endogenous upward instability of normalized financial and goods flows, as well as stocks representing financial risk; (2) implications of financial flows for financial liabilities that are intermediated by a rising arrival rate λ of jumps that instantaneously bring (1) reduced financial obligations L, (2) lower capacity utilization u, and (3) more-conservative (lower) borrowing b. These reductions set the stage for another financial-and-capacity-utilization boom. We have called the endogenous effects on λ of accumulated obligations L *probabilistic stock-flow consistency.* Perhaps most notably, of the key elements in Minsky's model of financial instability (e.g., Minsky 1986), this paper has omitted effects of countercyclical government policies that help to restore and sustain renewed growth and sometimes inflation. The paper has sought to rigorously and carefully state a Poisson-Minsky model in a relatively simple setting to lay compactly the groundwork for more elaborate applications.

VIII. Appendix

a. Computing the time derivative of L

To compute the differential equation for L as in Section II, we start with this equation

$$L = \int_{t_0}^{t} b\, ds \quad (1)$$

and differentiate w.r.t. the upper limit of integration (2)

For the SFC model of Section VI, we used the debt integral

$$L = \int_{t_0}^{t} b - \rho L\, ds \quad (1')$$

In this case, the differential equation for L turns out to be

$$\dot{L} = b - \rho L$$

b. The Stationary Equilibrium

We used the following MATLAB® program to compute an analytical solution for a stationary equilibrium of the model of Section 7.

```
syms a c d a_b0 a_u0 a_bL a_uL b u L ir rho solb solu solL;

eqns = [0 == a*u - a_bL*(ir+rho)*L + a_b0, 0 == c*b + d*u - a_uL*(ir+rho)*L + a_u0, 0 == b - rho*L];

vars = [b u L];

[solb, solu, solL] = solve(eqns, vars)
```

MATLAB® yielded the following solution for the three variables b, u, and L respectively.

```
>> Minimal_MInsky_Model_reduced_equil_corr

solb = -(rho*(a*a_u0 - a_b0*d))/(a_bL*d*ir - a*a_uL*ir - a*a_uL*rho + a*c*rho + a_bL*d*rho)

solu = -(a_bL*a_u0*ir - a_b0*a_uL*ir - a_b0*a_uL*rho + a_bL*a_u0*rho + a_b0*c*rho)/(a_bL*d*ir - a*a_uL*ir - a*a_uL*rho + a*c*rho + a_bL*d*rho)

 solL =-(a*a_u0 - a_b0*d)/(a_bL*d*ir - a*a_uL*ir - a*a_uL*rho + a*c*rho + a_bL*d*rho)
```

As the stationary equilibrium is not stable or economically meaningful (see Section II), we have included this calculation here only for the sake of completeness. We will omit a similar computation for the revised model presented in Section VI.

c. Computing the roots of the simplified system in symbols

To obtain the eigenvalues for the simplified model, I used the following program in MATLAB® R2018a with Symbolic Toolbox, which I mentioned in Section II.

```
syms a b c d;

J2 = [0 a 0; c d 0; 1,0,0];

eigenvreduc = eig(J2)
```

The program generated the following output:

```
eigenvreduc =


                     0

 d/2 - (d^2 + 4*a*c)^(1/2)/2

 d/2 + (d^2 + 4*a*c)^(1/2)/2
```

d. Stability conditions for the more-SFC model of Section VI

In Section VI, we use a numerical example that is unstable, with a pair of complex roots. We can find that adding terms that increase SFC feedbacks from private-sector debt not surprisingly bring in the possibility of a *stable* linear system. In fact, the system is stable under a set of three conditions based on conditions on the coefficients of the characteristic polynomial, which is of order three.

The author used the following MATLAB® program with a symbolic math toolbox:

```
syms a_bu a_ub a_uL a_uu a_bL iplusrho rho;

Jsfc = [0 a_bu -a_bL*iplusrho; a_ub a_uu -a_uL*iplusrho; 1 0 -rho];

charpolysfc = charpoly(Jsfc)
```

The package reported coefficients for a normalized polynomial as follows.

a_0 = 1

a_1 =rho - a_uu

a_2 = a_bL*iplusrho - a_bu*a_ub - a_uu*rhoa_3 = a_bu*a_uL*iplusrho - a_bL*a_uu*iplusrho - a_bu*a_ub*rho

From Gandolfo (1997, 221), a set of necessary and sufficient conditions for the real parts of all roots to be negative and hence for the system to be stable, given that one has normalized the characteristic polynomial so that $a_0 = 1$, as in the above:

(1) $a_1 > 0$

(2) $a_3 > 0$

(3) $a_1 a_2 – a_3 > 0$

Starting with any set of parameters for which the stability conditions are met, one can change them so that the system is unstable. For example, one could a_uu large enough that a_1 = rho - a_uu < 0, resulting in a failure to meet condition (1). We know that ρ < 1, while a_uu depends on the usual considerations related to the conditions for Keynes-Kaldor-Kalecki output-adjustment stability.

The numerical example in Section VI illustrates the exact computation of eigenvalues for a model with such feedbacks.

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